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ROBERT E. HORTON, *Editor*

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ORBIT TRANSFER GROUPS

LOUIS G. VARGO, Costa Mesa, California

1. Introduction. Analysis of the physical process of transferring a space vehicle from one orbit to another defines an important area of astrodynamic investigation. Certain classes of orbit transfer can be shown to have transformation group representations. The resultant greater insight into the mathematical structure may serve to unify and extend present methods and findings.

The terminology and symbolism contained in [1] will be used. Keplerian orbits are completely specified by a set of values for six independent variables x_i—the "elements." A transformation to x_i' results from selection of parameter values a_r $(r=1, 2, 3, 4)$ corresponding to three components of a velocity impulse applied at some point on x_i. Continuous functions

$$x_i' = f_i(x_1, \cdots, x_6; a_1, \cdots, a_4) \equiv f_i(x, a), \qquad (i = 1, \cdots, 6) \tag{1}$$

exist, but the operator in $T_a x = x'$ does not possess the closure property in general. If the product $U_{ab} = T_b T_a$ is taken to be the generic operator, then $U_{ab} U_{cd}$ is not unique. Thus, the general one- and two-impulse transformation classes do not form groups and the question of exposing all sub-classes which satisfy the group conditions is perhaps the central problem. One example of such a subclass will be given in this brief note.

2. An Example. Consider the transformation of an orbit produced by application of a tangential velocity impulse at a fixed point on the orbit as shown in Figure 1. Only one parameter, the impulse magnitude, is involved. Let the

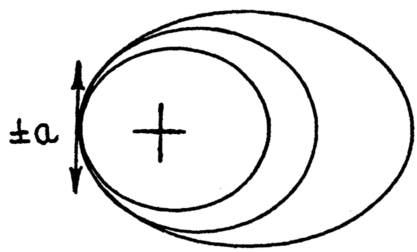


FIG. 1.

point be the periapsis, that is, the point of minimum focal distance on an elliptic orbit. This class of transformations is a group, as are all fixed-point one-impulse transfers, since vector addition at a point displays the group property. The one-parameter group defined has interest because of its physical significance [2] and mathematical equivalence to the unidirectional translation group.

Only the size (semi-major axis) and shape (eccentricity) of the ellipse are changed. Let the variables x_1 and x_2, however, respectively denote the periapsidal and apoapsidal speed in the orbit. These quantities are simply-related to the semi-major axis and eccentricity. Using energy and momentum conservation principles, the transformation equations are

$$x_1' = x_1 + a, \tag{2.1}$$

$$x_2' = \frac{x_1x_2 - 2ax_1 - a^2}{x_1 + a}. \tag{2.2}$$

It may be readily verified that $T_bT_a = T_{a+b}$. The "trajectories" of the group are solutions of the differential equation

$$\frac{dx_1'}{\xi_1} = \frac{dx_2'}{\xi_2}, \tag{3}$$

where $\xi_1 = dx_1'/da = 1$ and

$$\xi_2 = \frac{dx_2'}{da} = -2 - \frac{x_2'}{x_1'}.$$

The symbol or basis of the group is

$$Xf = \frac{\partial f}{\partial x_1} - \left(2 + \frac{x_2}{x_1}\right)\frac{\partial f}{\partial x_2}.$$

From (3), the translation-group representation becomes (2.1) and $x_3' \equiv x_1'(x_1' + x_2') = x_1(x_1 + x_2) \equiv x_3$. Thus the product of the periapsidal speed and the sum of the two apsidal speeds is invariant under periapsis impulse applications.

References

1. Eisenhart, L. P. *Continuous Groups of Transformations*, Princeton University Press, Princeton, 1933.
2. Vargo, L. G. *A Generalization of the Minimum-Impulse Theorem to the Restricted Three-Body Problem*, Journal of the British Interplanetary Society, vol. 17, no. 5, Sept.–Oct. 1959, p. 124.

FOUR MATHEMATICAL SURPRISES

The late NORMAN ANNING

I. $(1+i)(1+2i)(1+3i) = (1-i)(1-2i)(1-3i)$
Indeed, simplification gives -10.

II. $39 = 6 \times 6.5$, and $(6+6.5)/2 = 6.25$
$39 = 6.25 \times 6.24$, and $(6.25+6.24)/2 = 6.245$
$(6.245)^2 = 39.000025$

III. Six permutations of the set of digits 000111222 are square numbers. Thus:

$100220121 = (10011)^2$	$001022121 = (01011)^2$
$102212100 = (10110)^2$	$001212201 = (01101)^2$
$121022001 = (11001)^2$	
$121220100 = (11010)^2$	

Admittedly, the two permutations on the right are not very reputable nine-digit squares.

IV. The roots of $X^2 - 68X + 999.9999 = 0$ are rational. This follows from the factorization,

$$(X - 46.49)(X - 21.51) = 0$$

PROJECTIVE METHODS IN EUCLIDEAN GEOMETRY

SEYMOUR SCHUSTER, Carleton College

The utilization of the methods of projective geometry for gaining insight and for actually solving problems of Euclidean geometry is as old as projective geometry itself. However, a recently published solution to a problem on the William Lowell Putnam Competition indicated to me that, although projective methods are old, and familiar to geometers, they are not as widely known as they should be among sophisticated collegians. The given solution was analytic and quite cumbersome, while the projective solution is . . . well, let the reader decide for himself.

Problem: (See Figure 1.) Given any acute-angled triangle ABC and one altitude AH, select any point D on AH, then draw BD and extend it until it intersects AC in E. Draw CD and extend it until it intersects AB in F. Prove angle AHE = angle AHF.

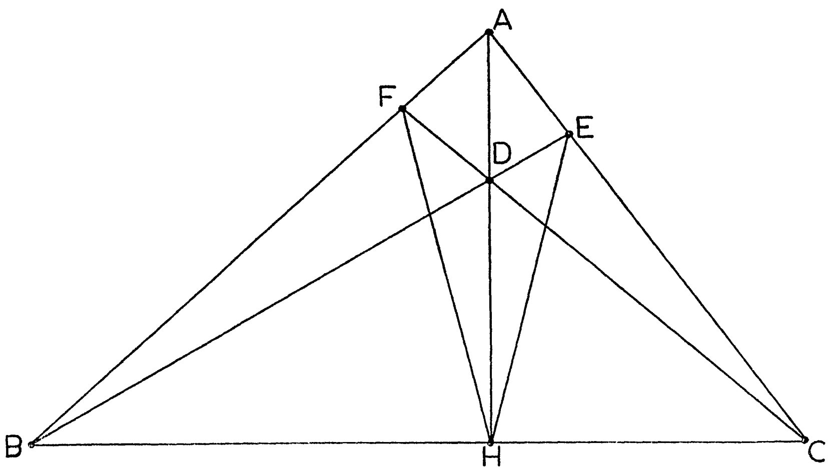


FIG. 1

Projective Solution: Lines AB, BD, DC, and CA are the sides of a quadrilateral, which implies that HE is the harmonic conjugate of HF with respect to HA and BC. But since AH is perpendicular to BC, angle AHE = angle AHF.

Actually, the projective geometric conclusion from the harmonic set $H(HA, BC; HE, HF)$ may be stated more generally: Lines HA and BC are the bisectors of the angles formed by lines HE and HF. Thus, we see that the hypothesis of "acute-angled" is stronger than necessary. As long as HA is interior to angle EHF, the result holds (see Figure 2); otherwise BC is the bisector of angle EHF, with the result that angle BHE = angle BHF (see Figure 3), and of course HA is the other bisector of the angles formed by lines HF and HE.

Our second example is also from the Putnam Competition.

Problem: Given a parabola, exhibit a Euclidean (straight edge and compass) construction which locates the focus and directrix.

Solution: A look at the parabola from the projective point of view will yield enough insight to give us a start. A parabola is a conic tangent to the line at

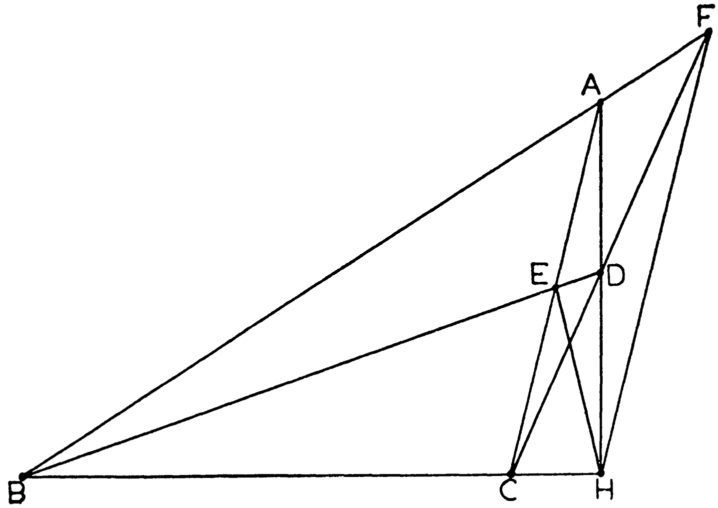


FIG. 2

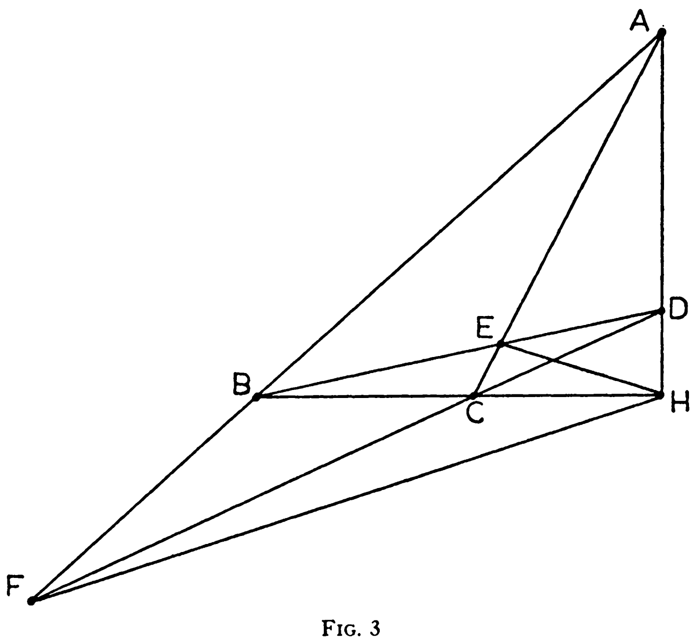


FIG. 3

infinity (see Figure 4). The polar p of any point P on the line at infinity bisects the chords which pass through P; this follows from the fact that the midpoint A of chord A_1A_2 is the harmonic conjugate of the point at infinity (on A_1A_2) with respect to A_1 and A_2. Further, p is parallel to the axis of the parabola, and the tangent at the point where p meets the parabola is parallel to A_1A_2. Thus, we have our beginning.

Construct two parallel chords A_1A_2 and B_1B_2. Call their midpoints A and B, respectively. Then AB is parallel to the axis, and the tangent t at the inter-

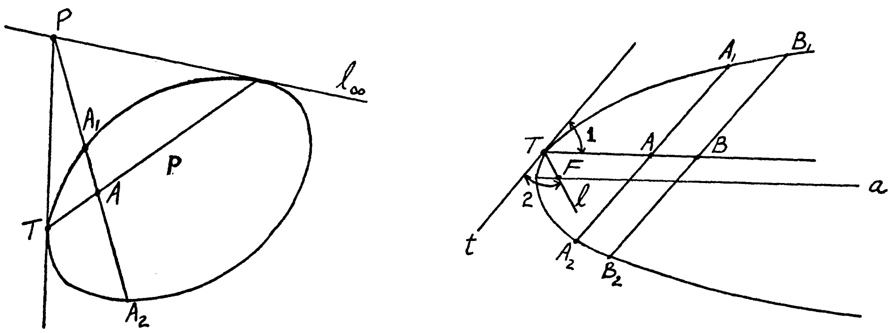
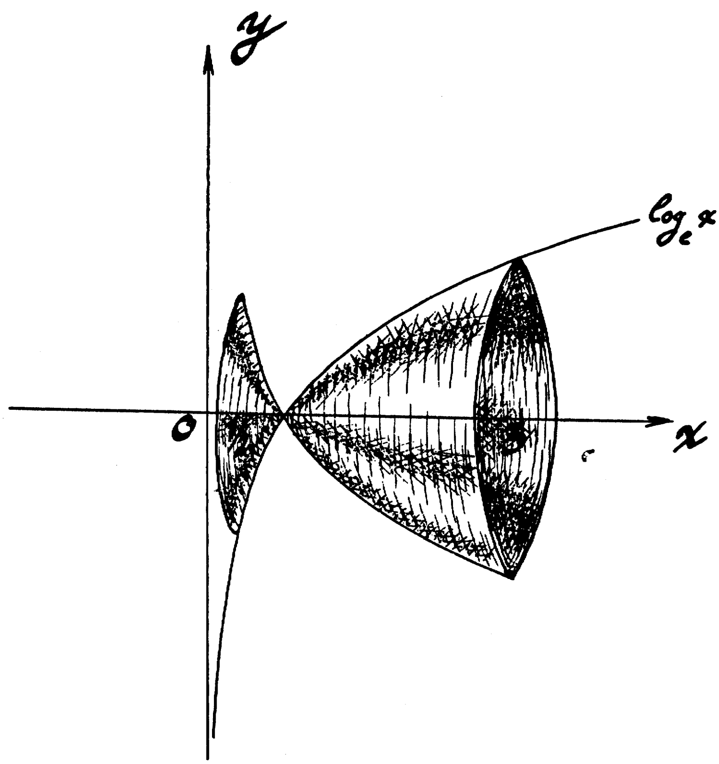


FIG. 4

section T of AB with the parabola is simply parallel to A_1A_2. Now Euclidean considerations may easily take over.

The axis can be constructed by the following procedure. Construct chord C_1C_2 perpendicular to AB, and locate the midpoint C of C_1C_2. The axis a is then the line through C perpendicular to C_1C_2 (i.e. parallel to AB).

By using the focal property of the parabola, the focus F can be located as follows. Construct the line l, so that the angle of reflection, $\angle 2$, equals the angle of incidence, $\angle 1$, at T. The intersection of l and a is F. The directrix is now a trivial matter.



The surface obtained by rotating $\log_e x$, $\frac{1}{3} \leq x \leq 3$ about the x-axis.

THE APPEARANCE OF FIBONACCI NUMBERS AND THE Q MATRIX IN ELECTRICAL NETWORK THEORY

S. L. BASIN, San Jose State College, California

1. Introduction. The purpose of this paper is to demonstrate where certain topics in pure mathematics may appear in the analysis of physical systems; it is not intended to develop new techniques in the analysis of two-terminal-pair networks. In a previous paper [1] I showed where Fibonacci numbers appear in the characterization of a particular electrical network. We shall now approach electrical networks with a little more generality and show where the Fibonacci numbers, along with the related Q matrix, evolve from pure circuit theory. The eigenvalues of the Q matrix appear as *image impedances* of a two-terminal-pair network. Some familiarity with a few techniques developed by the electrical engineer will be useful before proceeding into the final results.

2. A Prefatory Note on Circuit Analysis. Background information relating to circuit analysis can be found in most elementary electrical engineering texts; an excellent treatment is given by Skilling [2]. When Kirchhoff's voltage law is applied to an electrical network having L loops the following L simultaneous equations result:

$$\begin{aligned}
Z_{11}I_1 + Z_{12}I_2 + \cdots + Z_{1L}I_L &= E_1 \\
Z_{21}I_1 + Z_{22}I_2 + \cdots + Z_{2L}I_L &= E_2 \\
Z_{31}I_1 + Z_{32}I_2 + \cdots + Z_{3L}I_L &= E_3 \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
Z_{L1}I_1 + Z_{L2}I_2 + \cdots + Z_{LL}I_L &= E_L.
\end{aligned} \tag{1}$$

I_j denotes the loop current in loop j, E_j denotes the total electromotive force (emf) in loop j and $Z_{11}, Z_{22}, \cdots, Z_{LL}$ are the self impedances in each loop. The impedances Z_{ij} and Z_{ji} are the mutual impedances and by definition are the ratios of the voltage in a loop to the current in another loop that produced the voltage. In this paper we will be concerned with linear networks that are passive and bilateral. The term bilateral refers to coupling between circuits (loops) through mutual elements. If a current in one loop produces a voltage in some other loop and if the same current in the second loop produces the same voltage in the first loop then the coupling is symmetrically bilateral. Since the circuits that we will be discussing are bilateral we have, in general, $Z_{ij}=Z_{ji}$, i.e. current in loop j will produce the same voltage in loop i that equal current in loop i would cause in loop j. This fails for circuit elements that are not bilaterally symmetrical, e.g. vacuum tubes.

Two physical systems of different nature, e.g. electrical and mechanical, are called analogs if their operations are described by equations of the same form. A form of analogy called *duality* exists in nature and makes life easier for the analyst. If two physical systems of the same nature, i.e. two electrical networks, have equations of the same form but are physically different they are called *duals*. Hence, the dual of the network described by (1) is easily described by applying Kirchhoff's current law to this dual having N nodes. N simultaneous equations in N unknowns result:

$$\begin{aligned} Y_{aa}V_a + Y_{ab}V_b + Y_{ac}V_c + \cdots + Y_{an}V_n &= I_a \\ Y_{ba}V_a + Y_{bb}V_b + Y_{bc}V_c + \cdots + Y_{bn}V_n &= I_b \\ Y_{ca}V_a + Y_{cb}V_b + Y_{cc}V_c + \cdots + Y_{cn}V_n &= I_c \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_{na}V_a + Y_{nb}V_b + Y_{nc}V_c + \cdots + Y_{nn}V_n &= I_n. \end{aligned} \tag{2}$$

Y_{aa}, Y_{bb}, $\cdots$, Y_{nn} are called the self-admittances at nodes a, b, $\cdots$, n respectively. Y_{ab} is the mutual admittance between nodes a and b. Both Y_{ab} and Y_{ba} are the sum of all admittances connected directly between nodes a and b, but prefixed with a negative sign. I_a is the source current flowing *toward* node a.

3. Solution of Loop Equations. Applying Cramer's rule to (1) we solve for the loop currents,

$$I_1 = \frac{N_1}{D},\ I_2 = \frac{N_2}{D},\ I_3 = \frac{N_3}{D},\ \cdots,\ I_j = \frac{N_j}{D},$$

where

$$D = \begin{vmatrix} Z_{11} & \cdots & Z_{1L} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ Z_{L1} & \cdots & Z_{LL} \end{vmatrix} \quad \text{and} \quad N_1 = \begin{vmatrix} E_1 & Z_{12} & \cdots & Z_{1L} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ E_L & Z_{L2} & \cdots & Z_{LL} \end{vmatrix}, \text{ etc.}$$

Partially expanding each numerator into its cofactors we may express the loop currents as,

$$\begin{aligned} I_1 &= \frac{N_1}{D} = \frac{\Delta_{11}}{D}E_1 + \frac{\Delta_{21}}{D}E_2 + \frac{\Delta_{31}}{D}E_3 + \cdots + \frac{\Delta_{l1}}{D}E_L \\ I_2 &= \frac{N_2}{D} = \frac{\Delta_{12}}{D}E_1 + \frac{\Delta_{22}}{D}E_2 + \frac{\Delta_{32}}{D}E_3 + \cdots + \frac{\Delta_{l2}}{D}E_L \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ I_j &= \frac{N_j}{D} = \frac{\Delta_{1j}}{D}E_1 + \frac{\Delta_{2j}}{D}E_2 + \frac{\Delta_{3j}}{D}E_3 + \cdots + \frac{\Delta_{lj}}{D}E_L \end{aligned} \tag{3}$$

where Δ_{ij} is the cofactor of the Z_{ij} term in D. These equations are greatly simplified by defining the following terms which turn out to have interesting physical significance.

$$\begin{aligned} y_{11} &= \frac{\Delta_{11}}{D}, \quad y_{12} = \frac{\Delta_{21}}{D}, \quad y_{13} = \frac{\Delta_{31}}{D}, \cdots \\ y_{21} &= \frac{\Delta_{12}}{D}, \quad y_{22} = \frac{\Delta_{22}}{D}, \quad y_{23} = \frac{\Delta_{32}}{D}, \cdots. \end{aligned} \tag{4}$$

The physical meaning of y_{ij} is the following. If $i=j$ then y_{ij} are called the *driving-point admittances.* If $i\neq j$ then y_{ij} are the *transfer admittances.* Driving-point admittance is the ratio of current in a given loop to the emf in the same loop when there are no sources of emf in any of the other loops. Transfer admittance is the ratio of current in some other loop to the driving emf. Driving-point and transfer impedances, z_{ij}, are defined in the solution of (2) in a similar manner.

4. Two-Terminal-Pair Networks. These are also known as coupling networks, fourpoles or two-port nets. The two-port nets considered in this paper will be restricted in the following manner. We first require that the circuit elements be linear for ordinary methods of analysis to apply and we further require that the two-port net be bilateral. In essence, our two-port nets are simply "black-boxes" with an input pair and an output pair of terminals, Fig. 1. A two-port net is not a general four-terminal network since it is restricted by requiring that the current at one terminal of a pair be equal and opposite the current at the other terminal of the pair at every instant of time. Loop and node equations can be written for the network in Fig. 1 regardless of the contents of the "black-

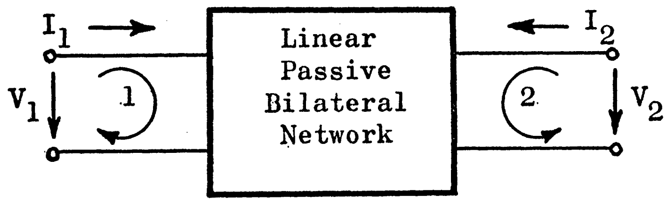


FIG. 1. The two-port net shown with reference directions for current and voltage at the input and output.

box." Since the network within the "black-box" is given as a passive network no sources exist within the box. Using the method outlined in section 2, we shall write the loop equations in matrix form, denoting the input and output loops by 1 and 2 respectively.

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{5}$$

The solution is given by,

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{6}$$

In a bilateral network $y_{ij}=y_{ji}$, the proof of this is quite elementary since all that is required is to show $\Delta_{ji}=\Delta_{ij}$ using properties of determinants. If Loop 2 of Fig. 1 is shorted the following relations result:

$$I_1 = y_{11}V_1 \rightarrow y_{11} = \frac{I_1}{V_1}\,.$$

$$I_2 = y_{21}V_1 \rightarrow y_{21} = \frac{I_2}{V_1}\,.$$

Short circuiting Loop 1 results in:

$$I_2 = y_{22}V_2 \rightarrow y_{22} = \frac{I_2}{V_2}.$$

$$I_1 = y_{12}V_2 \rightarrow y_{12} = \frac{I_1}{V_2}.$$

Then y_{ij}, when $i=j$, are the short-circuit driving-point admittances; y_{ij}, when $i\neq j$, are called the short-circuit transfer admittances. Considering the dual of Fig. 1, node equations result; their solution is given by:

$$\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned} \tag{7}$$

where $z_{ij}=\Delta_{ji}/|Y|$. These are the open-circuit driving point and open-circuit transfer impedances.

5. The Transmission Problem in General. The generalized transmission problem is not quite as simple as equations (6) and (7) may lead the mathematician to believe, since it is quite unusual that voltages at both ends of a two-port net are given simultaneously. We really want to develop an expression which would yield the voltage and current at one end of the network when given the voltage and current at the other end. We seek solutions to the following equations:

$$V_{\text{in}} = AV_{\text{out}} + BI_{\text{out}}. \tag{8a}$$

$$I_{\text{in}} = CV_{\text{out}} + DI_{\text{out}}. \tag{8b}$$

In order to solve for the network parameters $ABCD$ explicitly in terms of y_{ij} we must compare the reference directions in Fig. 2 with those in Fig. 1 and write

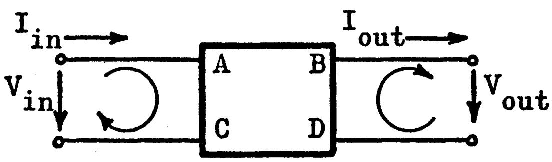


FIG. 2. The two-port net characterized by the network parameters ABCD.

the loop equations for Fig. 2 which are similar to (5) and (6). Using (6) and observing a change in reference direction in Fig. 2, we have,

$$I_{\text{in}} = y_{11}V_{\text{in}} + y_{12}V_{\text{out}}. \tag{9}$$

$$-I_{\text{out}} = y_{21}V_{\text{in}} + y_{22}V_{\text{out}}. \tag{10}$$

Solving for V_{in} in (10) and comparing with (8a) we have,

$$A = -\frac{y_{22}}{y_{21}} \qquad B = -\frac{1}{y_{21}}. \tag{11}$$

Substituting V_{in} into (9) and solving for I_{in}, comparing the result with (8b) will reveal,

$$C = \left(y_{12} - \frac{y_{11}y_{22}}{y_{21}}\right) \qquad D = -\frac{y_{11}}{y_{21}}. \tag{12}$$

A similar procedure gives the parameters $ABCD$ in terms of the open-circuit impedances,

$$A = \frac{z_{11}}{z_{21}} \qquad B = \frac{z_{11}z_{22} - z_{12}z_{21}}{z_{21}}$$

$$C = \frac{1}{z_{21}} \qquad D = \frac{z_{22}}{z_{21}}. \tag{13}$$

Since the general network functions $ABCD$ are functions of frequency, they are constant only at a constant frequency. After expressing $ABCD$ in terms of z_{ij} and evaluating the determinant of the 2×2 matrix we find,

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = AD - BC = \frac{z_{12}}{z_{21}} \equiv 1.$$

Since we are restricting ourselves to passive bilateral networks, $z_{ij} = z_{ji}$ and the determinant of the above 2×2 matrix is identically one. This result will be useful later.

So far we have shown how any two-port net is characterized by any one of three 2×2 matrices,

$$Y = \begin{pmatrix} y_{12} & y_{12} \\ y_{21} & y_{22}\end{pmatrix}, \qquad Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{pmatrix}, \qquad \begin{pmatrix} A & B \\ C & D\end{pmatrix}.$$

The admittance matrix Y was developed by writing the loop equations and the impedance matrix Z evolved from the node equations of the network. Either one of these three matrices is sufficient to completely describe the two-port net, in fact, the y_{ij}'s can be shown to depend on the z_{ij}'s [2]. Several other types of hybrid transfer functions exist; however, these will not be of value here.

6. Two-Port Nets in Cascade. Before considering electrical networks of particular interest (those having transfer matrices whose elements are Fibonacci numbers) we shall demonstrate how the three matrices, described above, are

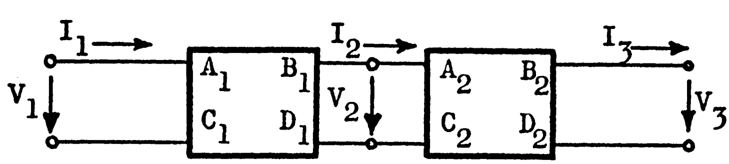


FIG. 3. Two two-port nets in cascade.

utilized when cascading any number of two-port nets. Denote the input vector by x, the output of the first fourpole by s and the output of the second fourpole by t; then,

$$x = \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} \qquad s = \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} \qquad t = \begin{pmatrix} V_3 \\ I_3 \end{pmatrix}.$$

Since

$$x = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} s \quad \text{and} \quad s = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} t$$

then

$$x = \begin{pmatrix} A_2 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ A_2 & D_2 \end{pmatrix} t.$$

Denoting these matrices by K_i, and continuing in this manner, we find the transfer matrix for n two-port nets in cascade,

$$x = (K_1)(K_2)(K_3) \cdots (K_n)t. \tag{14}$$

Formulae for two-port nets in series or in parallel are given by Cauer [3] and Guillemin [4]; the results are as follows:

SERIES. The series connection of two two-port nets with impedance matrices $Z^{(1)}$ and $Z^{(2)}$ yields the impedance matrix for the equivalent two-port net in Fig. 4.

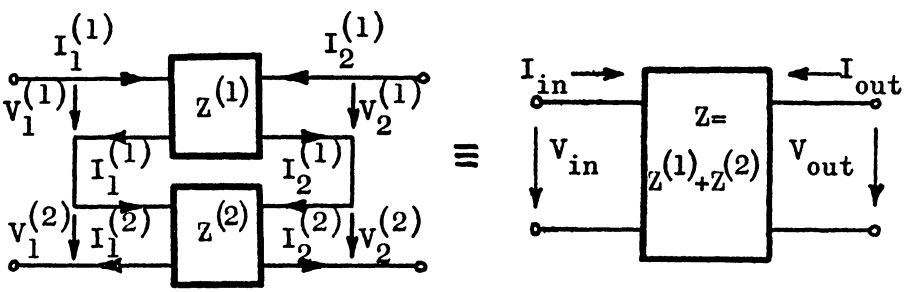


FIG. 4. Two two-port nets in series shown with the equivalent two-port net.

PARALLEL. The parallel connection of two two-port nets has an equivalent two-port net whose admittance matrix is the sum of the original admittance matrices; hence, $Y = Y^{(1)} + Y^{(2)}$.

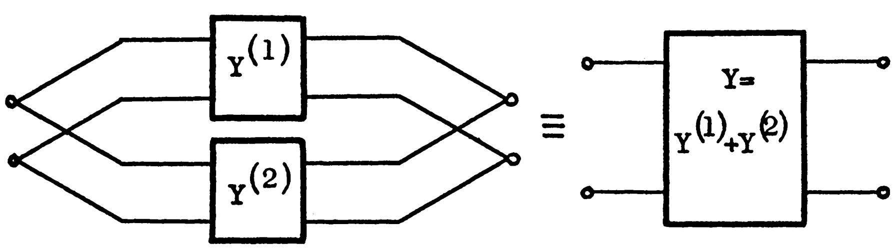


FIG. 5. Two two-port nets in parallel shown with the equivalent two-port net.

7. The Q Matrix. This matrix is a 2×2 matrix whose elements are certain Fibonacci numbers; it is defined as,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

By induction,

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

where F_n is the nth Fibonacci number defined by the recursion relationship $F_n = F_{n-1} + F_{n-2}$ $(n \geqq 3)$. If we synthesize a two-port net such that,

$$Z = Q^n, \qquad Y = Q^n \quad \text{or} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = Q^n$$

several interesting properties of this network follow from the study of the Q^n matrix. Consider a two-port net such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Notice that this network may not be symmetrical in general. Since $A = D$ for symmetrical two-port nets, we would have $F_{n+1} = F_{n-1}$ this only holds for $n = 0$ in which case the transfer matrix would be the identity matrix.

Cascading two two-port nets, as in Fig. 3, whose transfer matrices are both Q^n we have,

$$x = Q^{2n}t \tag{15}$$

The Q matrices form an Abelian group with matrix multiplication as the binary combination. Hence, we may easily determine the output vectors from (15) since,

$$t = Q^{-2n}x. \tag{16}$$

8. Using Properties of the Q Matrix. We shall define an output vector for the network in Fig. 3 such that

$$t_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}. \tag{17}$$

We call (17) the Fibonacci vector t_n. In general,

$$Q^m t_n = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_{n+m+1} \\ F_{n+m} \end{pmatrix} = t_{n+m}. \tag{18}$$

Therefore, the Q matrix maps t_n into t_{n+1} and cascading m two-port nets we have t_n mapped into t_{n+m}. Cascading two-port nets whose transfer matrix is Q^2 we map t_1 into t_3, t_3 into t_5 and by successive additions of two-port nets to the original network we may generate all the vectors of the form t_{2n+1}, Fig. 6. Similarly, taking the output vector t_0 we generate all input vectors of the form t_{2n}.

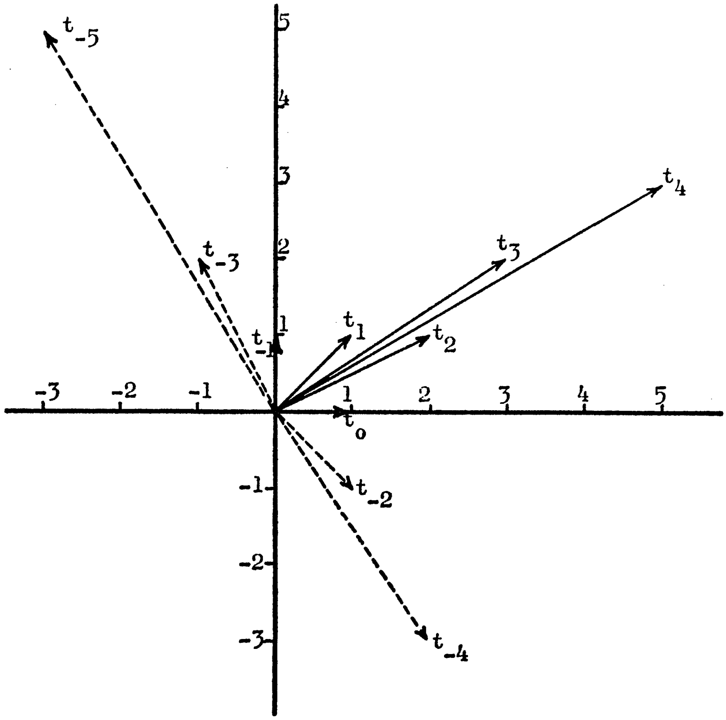


FIG. 6. Some of the Fibonacci vectors in the space R^2.

We are generally interested in knowing the output in terms of the imput; however, we shall defer this until we investigate the linearity of Q^{-2m}. If

$$r_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \text{then} \quad \alpha r_1 + \beta r_2 = \begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{pmatrix}$$

and

$$Q^{2m}(\alpha r_1 + \beta r_2) = \begin{pmatrix} F_{2m+1} & F_{2m} \\ F_{2m} & F_{2m-1} \end{pmatrix} \begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{pmatrix} = \alpha Q^{2m} r_1 + \beta Q^{2m} r_2$$

$$Q^{2m}(\alpha r_1 + \beta r_2) = \alpha Q^{2m} r_1 + \beta Q^{2m} r_2 \tag{19}$$

therefore Q^{2m} is a linear transformation. Similarly, the linearity of Q^{-2m} follows.

$$Q^{-2m}(\alpha r_1 + \beta r_2) = \alpha Q^{-2m} r_1 + \beta Q^{-2m} r_2. \tag{20}$$

Now that the linearity of Q^{-2m} has been established, an interesting result follows. The Fibonacci vectors t_0 and t_{-1} form an orthonormal basis in R^2, Fig. 6; any input vector may be written as,

$$x = \alpha t_0 + \beta t_{-1}. \tag{21}$$

Using (16), (20) and (21) we have,

$$t = Q^{-2m} x = \alpha Q^{-2m} t_{-1} + \beta Q^{-2m} t_{-1}. \tag{22}$$

The matrix Q^{-2m} is obtained by letting $n=-2m$ in the expression formerly defined for Q^n; we must also recall, $F_m=(-1)^{m+1}F_{-m}$ which defines the Fibonacci numbers for negative subscripts. Therefore, (22) becomes,

$$t = Q^{-2m}x = \begin{pmatrix} \alpha F_{-2m+1} + \beta F_{2m} \\ \alpha F_{2m} + \beta F_{2m+1} \end{pmatrix} = \begin{pmatrix} \alpha F_{2m-1} + \beta F_{2m} \\ \alpha F_{2m} + \beta F_{2m+1} \end{pmatrix}.$$

This result is simplified by observing that the elements of the output vectors, in (23), are simply consecutive terms of the generalized Fibonacci sequence, $H_n=H_{n-1}+H_{n-2}$, where $H_1=\alpha$, $H_2=\beta$, we have

$$\{\alpha, \beta, \alpha+\beta, \alpha+2\beta, 2\alpha+3\beta, 3\alpha+5\beta, \cdots, \alpha F_{n-1}+\beta F_n, \cdots\}.$$

Therefore, successive application of the Q^{-2m} matrix to an input vector $\binom{\alpha}{\beta}$ maps this vector into vectors whose elements are successive terms of the above generalized Fibonacci sequence.

9. Image Impedance. If an external impedance W is connected across the output terminals of a two-port net it is obvious that a change will occur in the impedance looking into the input terminals (driving-point impedance z_{11}). It will be convenient to determine a value of W such that the driving point impedance equals W. A network is said to be terminated in its image impedance if $z_{11}=W$.

Consider the following two-port net, Fig. 7, as an example. We treat this ladder network, consisting of nL-sections, as n two-port nets in cascade. Since

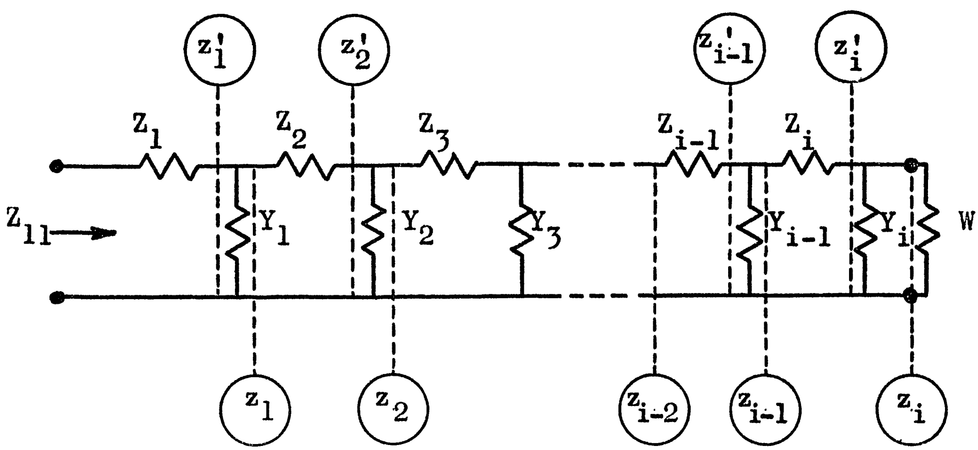


FIG. 7. The ladder network considered as a two-port net terminated with impedance W.

$V_{\text{out}}=WI_{\text{out}}$ and $z_{11}=V_{\text{in}}/I_{\text{in}}$ using (8a, b) we have,

$$z_{11} = \frac{AW+B}{CW+D}. \tag{24}$$

Therefore, z_{11} is a bilinear function of W. Setting $z_{11}=W$, W becomes the fixed point of bilinear transformation (24). If the transfer matrix of this ladder network is Q^{2n} then W equals the eigenvalues of Q, which will be demonstrated.

Start at the right hand side of Fig. 7 and determine the impedances of the partial networks looking to the right of each broken line marked z_i. In performing this calculation note that impedance is the reciprocal of admittance. At z_i the admittance is $1/W$; at z_i' the impedance is

$$\frac{1}{y_i + \dfrac{1}{W}}.$$

The impedance to the right of z_{i-1} is

$$z_i = \frac{1}{y_i + \dfrac{1}{W}}.$$

Continuing in the same way, the driving-point impedance z_{11} is,

$$z_{11} = z_1 + \cfrac{1}{y_1 + \cfrac{1}{z_2 + \cfrac{1}{y_2 + \cfrac{1}{z_3 + \cfrac{1}{y_3 + \cdots + \cfrac{1}{y_i + \cfrac{1}{W}}}}}}}. \tag{25}$$

We may express the continued fraction (25) in matrix form using a method developed by Frame [5]. The continued fraction (25) may be written in the form,

$$z_{11} = \frac{P_i + \left(\dfrac{1}{W}\right)S_i}{R_i + \left(\dfrac{1}{W}\right)T_i}$$

this is evident by (24). Now let M_i be defined as,

$$M_i = \begin{pmatrix} P_i & S_i \\ R_i & T_i \end{pmatrix}. \tag{26}$$

If M_1 is defined by,

$$M_1 = \begin{pmatrix} P_1 & S_1 \\ R_1 & T_1 \end{pmatrix} = \begin{pmatrix} z_1 & 1 \\ 1 & 0 \end{pmatrix}$$

and the recursion formulae developed by Frame are,

$$P_i = a_i P_{i-1} + P_{i-2}, \qquad S_i = P_{i-1}$$
$$R_i = a_i R_{i-1} + R_{i-2}, \qquad T_i = R_{i-1}$$

where the a_i are the quotients of the fraction (25) then (26) may be rewritten as,

$$M_i = \begin{pmatrix} P_i & P_{i-1} \\ R_i & R_{i-1} \end{pmatrix} = \begin{pmatrix} P_{i-1} & P_{i-2} \\ R_{i-1} & R_{i-2} \end{pmatrix} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix} = M_{i-1} \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}.$$

By induction, the continued fraction (25) becomes,

$$M_n = \begin{pmatrix} z_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} y_n & 1 \\ 1 & 0 \end{pmatrix}.$$

If the ladder network, Fig. 7, consists of impedances all of unit value then, $M_n = Q^n$.

The eigenvalues of the Q matrix are given by the characteristic equation, $\lambda^2 - \lambda - 1 = 0$. Solving for λ, we have the eigenvalues

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

Solving for the fixed points of (24) we find,

$$CW^2 + (D - A)W - B = 0$$

therefore,

$$W = \frac{(A - D) \pm \sqrt{(D - A)^2 + 4BC}}{2C}. \tag{27}$$

If Q^{2n} is the transfer matrix, (27) becomes,

$$W = \frac{1 \pm \sqrt{5}}{2}.$$

and W coincides with the eigenvalues of the Q matrix.

THEOREM 1. *Let A, B, C, D be integers satisfying $A > 0$, $D > 0$ and $AD - BC = 1$ and let the roots of $\lambda^2 - \lambda - 1 = 0$ be the fixed points of*

$$W = \frac{Az + B}{Cz + D}.$$

Then it is necessary and sufficient for all integral $n \neq 0$ that, $A = F_{2n+1}$, $B = C = F_{2n}$, $D = F_{2n-1}$ where F_n is the nth Fibonacci number.

Note, that it is sufficient to have

$$A = F_{n+1}, \qquad B = C = F_n, \qquad D = F_{n-1}$$

for $W=\lambda$, however, the requirement imposed on us by the bilateral network is that

$$AD - BC = 1. \tag{28}$$

The determinate of the Q^n matrix alternates in signs since

$$\det Q^n = (-1)^n$$

therefore n must be even to satisfy (28).

The image impedance is also known as the iterative impedance since it is the value of z_{11} for an infinite cascade of two-port nets. Note that the impedances $\textcircled{z_i}$ in Fig. 7 are the successive convergents to W.

10. Fibonacci Polynomials. An abundance of literature concerning the study of cascaded two-port networks, by methods other than the chain matrix method described above, has recently appeared in the literature [6], [7], [8], [9] and [10]. These papers suggest that a relationship exists between the Fibonacci sequence, hyperbolic functions, continuants and certain Jacobi polynomials. The specific relationships between the Chebyshev polynomials and the Fibonacci sequence has been studied by Basin [11]. In the analysis of a particular ladder network, A. M. Morgan-Voyce [6] defines a set of polynomials by:

$$\begin{aligned} b_0(x) &= 1, \qquad B_0(x) = 1 \\ b_n(x) &= xB_{n-1}(x) + b_{n-1}(x); \qquad n \geqq 1 \\ B_n(x) &= (x+1)B_{n-1}(x) + b_{n-1}(x); \qquad n \geqq 1. \end{aligned} \tag{29}$$

From (29) we observe that $b_n(x)$ and $B_n(x)$ both satisfy the same recurrence relationship, namely:

$$\begin{aligned} b_{n+1}(x) &= (x+2)b_n(x) - b_{n-1}(x); \qquad n \geqq 1 \\ B_{n+1}(x) &= (x+2)B_n(x) - B_{n-1}(x); \qquad n \geqq 1. \end{aligned} \tag{30}$$

The above polynomials possess many fascinating properties which have been discovered by V. E. Hoggatt, J. L. Brown, Jr. and S. L. Basin (in private communications). A few of these properties are described in the following theorems, easily proved by mathematical induction.

THEOREM 2.

$$B_n(1) = F_{2n+2} \quad \text{and} \quad b_n(1) = F_{2n+1}.$$

THEOREM 3. $b_{n+1}(x) = x\sum_{i=0}^{n} B_i(x) + b_0(x)$; $n \geqq 0$

$$B_{n+1}(x) = B_n(x) + b_{n+1}(x).$$

THEOREM 4. $b_n(0) = 1$ *for all* $n > 0$

$$B_n(0) = n+1 \text{ for all } n > 0.$$

THEOREM 5. *For n, any fixed integer,*

(i)
$$\lim_{x\to\infty} \frac{b_n(x)}{B_n(x)} = 1$$

(ii) $$\lim_{x\to 0} \frac{b_n(x)}{B_n(x)} = \frac{1}{n+1}\cdot$$

Each of the above sets of polynomials are Jacobi polynomials since $b_n(x)$ is orthogonal on $(-4, 0)$ and $B_n(x)$ is orthogonal on the same interval with respect to the weight functions,

$$W_b(x) = (-x)^{-1/2}(x+4)^{1/2} \qquad \text{for } b_n(x)$$

and

$$W_B(x) = (-x)^{1/2}(x+4)^{1/2} \qquad \text{for } B_n(x).$$

11. The Matrix Treatment of $b_n(x)$ and $B_n(x)$. The polynomials $B_n(x)$ can be generated by matrix multiplication if we define,

$$B = \begin{pmatrix} x+2 & -1 \\ 1 & 0 \end{pmatrix}.$$

By induction,

$$B^n = \begin{pmatrix} B_n(x) & -B_{n-1}(x) \\ B_{n-1}(x) & -B_{n-2}(x) \end{pmatrix}.$$

The "hybrid" matrix defined by,

$$H = \begin{pmatrix} x+1 & 1 \\ x & 1 \end{pmatrix}$$

generates both $B_n(x)$ and $b_n(x)$ since,

$$H^n = \begin{pmatrix} b_n(x) & B_{n-1}(x) \\ xB_{n-1}(x) & b_{n-1}(x) \end{pmatrix}.$$

The hybrid matrix H can be factored into the familiar Q matrix and a generalization of the Q matrix,

$$H = Q\cdot Q(x) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}.$$

The $Q(x)$ matrix generates another set of Fibonacci polynomials satisfying,

$$f_1(x) = 1, \qquad f_2(x) = x$$
$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x).$$

The characteristic equations of the above matrices can be used to derive the explicit forms,

$$B_n(x) = \frac{1}{\sqrt{x^2+4x}}\left[\left(\frac{x+2+\sqrt{x^2+4x}}{2}\right)^n - \left(\frac{x+2-\sqrt{x^2+4x}}{2}\right)^n\right]$$

$$f_n(x) = \frac{1}{\sqrt{x^2+4}}\left[\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n\right].$$

Furthermore, the reciprocal of the characteristic polynomials of the above matrices yield the generating functions for the related Fibonacci polynomial, i.e.

$$\frac{1}{\lambda^2-(x+2)\lambda+1} = \sum_{n=0}^{\infty} B_n(x)\lambda^n$$

$$\frac{1}{\lambda^2 - x\lambda - 1} = \sum_{n=0}^{\infty} (-1)^{n+1} f_n(x)\lambda^n$$

we also have,

$$\frac{1-\lambda}{\lambda^2-(x+2)\lambda+1} = \sum_{n=0}^{\infty} b_n(x)\lambda^n.$$

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A GENERALISED FORM OF A THEOREM ON INTEGER QUOTIENTS OF PRODUCTS OF FACTORIALS

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Introduction. In a previous paper [1], I have defined "generalised integers" as follows:

Suppose given a finite or infinite sequence $\{p\}$ of real numbers (generalised primes) such that

$$1 < p_1 < p_2 < p_3 < \cdots .$$

Form the set $\{l\}$ of all possible p-products, i.e., products

$$p_1^{v_1} p_2^{v_2} \cdots$$

where $v_1, v_2, \cdots$ are integers $\geqq 0$ of which all but a finite number are 0.

Call these numbers "generalised integers" and suppose that no two generalised integers are equal if their v's are different. Then arrange $\{l\}$ as an increasing sequence:

$$1 = l_1 < l_2 < l_3 < \cdots < l_n < \cdots .$$

L. J. Mordell, [2], has written on "Integer quotients of products of factorials." The aim of this note is to prove three theorems on this topic for generalised integers.

Define

$$l_n! = l_n l_{n-1} \cdots l_1,$$

and

$$[l] = \text{the number of } l \text{ numbers} \leqq l,$$

e.g.,

$$[l_n] = n.$$

THEOREM 1. *If*

$$[l_n] = [l_a] + [l_b] + [l_c] + \cdots [l_k]$$

the quotient

$$\frac{l_n!}{l_a! l_b! l_c! \cdots l_k!} \tag{1}$$

is a generalised integer.

Proof. In [1], I have proved that the exponent of the highest power of a generalised prime P which divides $l_n!$ is

$$\left[\frac{l_n}{P}\right] + \left[\frac{l_n}{P^2}\right] + \left[\frac{l_n}{P^3}\right] + \cdots .$$

Also if $l_m \leqq l_n$, then

$$\left[\frac{l_n}{l_m}\right] \geqq \left[\frac{l_a}{l_m}\right] + \left[\frac{l_b}{l_m}\right] + \cdots + \left[\frac{l_k}{l_m}\right]. \tag{2}$$

Let p be a prime factor of $l_n!$, and put into this inequality successively

$$l_m = p,\ p^2,\ p^3, \text{ etc.}$$

By addition we then obtain the inequality

$$\sum_i \left[\frac{l_n}{p^i}\right] \geqq \sum_i \left[\frac{l_a}{p^i}\right] + \sum_i \left[\frac{l_b}{p^i}\right] + \cdots + \sum_i \left[\frac{l_k}{p^i}\right]. \tag{3}$$

But this inequality just states that the exponent of the highest power of p dividing $l_n!$ is greater than or equal to the sum of the exponents of the highest power of p dividing each of the terms $l_a!$, $l_b! \cdots l_k!$

Hence the number (1) is a generalised integer.

THEOREM 2. *If d_r, $\epsilon\{l\}$, is the greatest common divisor of all the products of the $l_a, l_b, \cdots, l_k$ taken r at a time, then the quotient*

$$\frac{l_{n-r}!d_r}{l_a!l_b! \cdots l_k!} \tag{4}$$

is a generalised integer.

Proof. If d_r in (4) is replaced by the product of any r of $l_a, l_b \cdots l_k$, then (4) becomes a generalised integer from Theorem 1. Consider the quotient

$$\frac{l_{n-r}!}{l_a!l_b! \cdots l_k!}.$$

If it is an integer there is nothing to prove. If not, suppose

$$\frac{l_{n-r}!}{l_a!l_b! \cdots l_k!} = \frac{p}{q}$$

where $(p, q) = 1$.

Then q must divide the product of any r of $l_a, l_b, \cdots, l_k$. Hence q must divide d_r and so (4) is an integer.

THEOREM 3. *The quotient*

$$\frac{(l_n l_m)!}{l_n!(l_m!)^{[l_n]}} \tag{5}$$

is a generalised integer.

Proof. We need only show that

$$\sum_i \left[\frac{l_n l_m}{p^i}\right] - \sum_i \left[\frac{l_n}{p^i}\right] - [l_n] \sum_i \left[\frac{l_m}{p^i}\right] \geqq 0$$

for every prime p which divides $(l_n l_m)!$

Let r and s denote integers such that

$$p^r \leqq l_n < p^{r+1} \quad \text{and} \quad p^s \leqq l_m < p^{s+1}.$$

Then

$$\sum_i \left[\frac{l_n l_m}{p^i}\right] - \sum_i \left[\frac{l_n}{p^i}\right] - n\sum_i \left[\frac{l_m}{p^i}\right]$$

$$= \sum_{i=1}^{s} \left[\frac{l_n l_m}{p^i}\right] + \sum_{i=s+1}^{r+s} \left[\frac{l_n l_m}{p^i}\right] + \sum_{i=r+s+1}^{\infty} \left[\frac{l_n l_m}{p^i}\right] - \sum_{i=1}^{r} \left[\frac{l_n}{p^i}\right] - n\sum_{i=1}^{s} \left[\frac{l_m}{p^i}\right]$$

$$= \sum_{i=1}^{s} \left(\left[\frac{l_n l_m}{p^i}\right] - n\left[\frac{l_m}{p^i}\right]\right) + \sum_{i=1}^{r} \left(\left[\frac{l_n l_m}{p^{i+s}}\right] - \left[\frac{l_n}{p^i}\right]\right) + \sum_{i=r+s+1}^{\infty} \left[\frac{l_n l_m}{p^i}\right]$$

$$\geqq \sum_{i=1}^{s} \left(\left[\frac{l_n l_m}{p^i}\right] - n\left[\frac{l_m}{p^i}\right]\right) + \sum_{i=1}^{r} \left(\left[\frac{l_n p^s}{p^{i+s}}\right] - \left[\frac{l_n}{p^i}\right]\right)$$

$$\geqq 0$$

since

$$\left[\frac{l_n l_m}{p^i}\right] \geqq n\left[\frac{l_m}{p^i}\right].$$

Hence the number (5) is a generalised integer.

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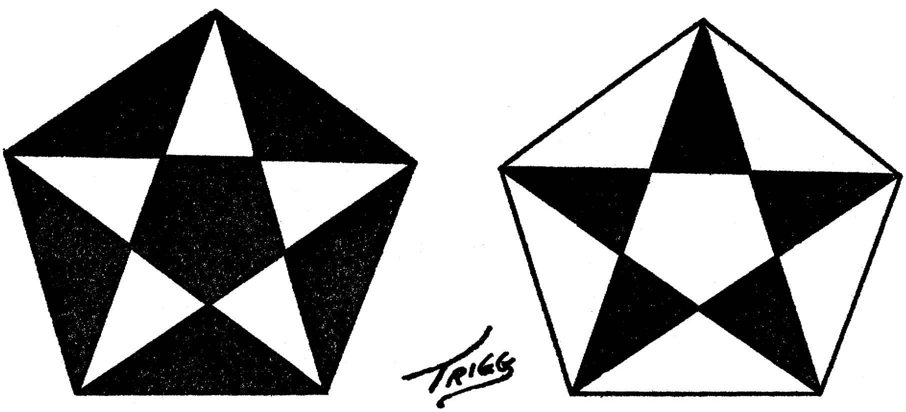
POSITIVE AND NEGATIVE I



EXTENDED TOPOLOGY: THE CONTINUITY CONCEPT

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Introduction. The concept of continuity as defined in topology and analysis has been extremely useful in discussing invariance of certain set-properties. However, the forms of the definitions are such as to make it seem that continuity is fruitfully confined to transformations or functions on spaces with an infinity of points. We have been developing a system of extended topology which includes concepts relevant to both finite and infinite spaces.

Applicability to finite spaces is necessary for a system of this kind to be basic since we want it to apply to numerical analysis, statistics, computing, and logic, as well as to language theory in general. Nearly ten years ago we had decided that any reasonable definition of continuity would necessarily display the homomorphisms of algebras as particular forms of continuity. However, we have only recently found a simple and intuitive way of defining continuity which includes topological and algebraic homomorphisms in the same framework.

Our feeling about the essential feature of continuity is that continuous mappings are those which preserve information or orderliness of some kind. However, this notion is too broad to implement now in some form of theory if we keep certain theorems from topology. Hence, in this paper we give a very general definition by standards of the current literature but do not attempt to carry out even a rudimentary classification of these kinds of continuity.

It is noticeable that while a function or transformation defined to be continuous maps elements into elements the invariant properties almost always refer to sets in the topological treatments. Thus connectedness, compactness, closedness or openness are all set properties. It is not necessary or useful always in dealing with invariance of set properties to consider set-valued transformations generated by element-valued ones. However, we will, in order not to deviate too far from custom, consider only such induced set-valued transformations here.

I. *A Simple Extension of Topological Continuity*

Let M be the space with null set N and let $\mathfrak{M}$ be the class of all subsets of M. Then a function u which associates with each subset of M a subset of M is called *expansive* provided $uX \supseteq X$ and $u(X_1 \cup X_2) \supseteq uX_1 \cup uX_2$ always. These expansive functions form a generalization of the closure functions of topology since an expansive function u in order to be a Kuratowski closure function must also be *idempotent* $(u(uX) \equiv uX)$, *additive* $u(X_1 \cup X_2) \equiv uX_1 \cup uX_2$ and satisfy $uN = N$. We have shown that much of the basic structure of topology is better considered for expansive functions than for the Kuratowski closure functions.

Now suppose M_1 is another space and v is an expansive function in M_1. Then if t: $M \rightarrow M_1$ is a function mapping M into M_1 this mapping induces a mapping from $\mathfrak{M}$ into $\mathfrak{M}_1$, the class of all subsets of M_1, by defining $tX = \{tp : p \in X\}$. We follow custom and use the same symbol t for the set-to-set transformation as for the original element-to-element one. Now if u is an expansive function in M, then t is said to be (u, v)*-continuous* provided $t(uX) \subseteq v(tX)$ for all X in $\mathfrak{M}$ or, more shortly, provided $tu \subseteqq vt$. In ordinary terms, if u and v are closure functions, then t is (u, v)-continuous provided the transform of the closure of X is contained in the closure of the transform of X. The reader may verify that this definition coincides with that of topology and hence of real or complex functions.

An ordered pair (X_1, X_2) of subsets of M is said to be *separated* with respect to an expansive function u in M provided $uX_1 \cap X_2 = N$; i.e., provided uX_1 and X_2 have no elements in common. This may be better read "X_2 is separated from X_1."

1. THEOREM. *A necessary and sufficient condition that t: $M \rightarrow M_1$ be (u, v)-continuous where u and v are expansive functions is that if $X_1, X_2 \in \mathfrak{M}$ and (X_1, X_2) is not separated with respect to u then (tX_1, tX_2) is not separated with respect to v.*

Remark. This theorem is proved in [3]. However, its proof follows from the definitions given and the reader may have the pleasure of verifying the conclusion himself. This is a *basic* theorem on several counts. It gives an image of a continuous mapping as one which does *not separate* ordered pairs of sets not separated in its domain. The theorem would *not* hold if we had blindly followed custom and required (X_1, X_2) to be considered separated provided $uX_1 \cap X_2 = N = X_1 \cap uX_2$ and it also would not then hold for general topology. Moreover, since separations of ordered set pairs are commonly in use which do not depend *in this fashion* on expansive or closure functions, we now have a springboard for generalization.

II. *What is a Separation?*

Let us consider pairs, (X_1, X_2), of sets from M. What sort of consideration might lead us to say X_2 is separated from X_1? *Certainly* we will not require that X_2 separated from X_1 will imply X_1 is separated from X_2, since many applications may be expected to be asymmetric. Shall we require $X_1 \cap X_2 = N$? No! For, X_1 and X_2 might have a common part considered trivial from some standpoint. By this reasoning and examples we were led to conclude that the only property we would use was that if (X_1, X_2) are separated and $X_1 \supseteqq X_3$, $X_2 \supseteqq X_4$ then (X_3, X_4) are separated. Thus we have a tentative picture of a separation as a *hereditary* binary relation in $\mathfrak{M}$; i.e., a separation is a set of ordered pairs of sets satisfying the above hereditary property.

But, wait! Could not there be something desirable added if we did not consider simply *pairs* of sets but also triples, quadruples, or sequences of sets as separated? It became clear that this also was reasonable and that for certain applications pairs of sets would not do. Hence we now may think if a separation as a hereditary subset of a sort of vector space in which each vector has sets as components. To be explicit let us assume S is a set of ordered ennuples, $\{(X_1, X_2, \cdots, X_n)\}$ of subsets of M which is hereditary. Then the set of ennuples which are not separated form an *ancestral* subset of the space of all ordered ennuples and this we call an *association* or the *association dual to S.*

Let A be an ennary association for M and B an ennary association for M_1. Then t is (A, B)-*continuous* provided $tA \subseteqq B$; i.e., $(X_1, \cdots, X_n) \in A$ implies $(tX_1, \cdots, tX_n) \in B$. *A continuous mapping is association preserving!*

To see that this definition is not foolish we now indicate its application to the characterization of homomorphisms. Let t map M into M_1 and suppose there is a binary operation $\cdot$ defined in M and a binary operation $\circ$ defined in M_1. Then t is a *homomorphism* (multiplication-preserving) provided $t(p_1 \cdot p_2) = (tp_1) \circ (tp_2)$ for all $p_1, p_2 \in M$.

We let A be the minimal ternary association in M including all triples of the form $(p_1, p_2, p_1 \cdot p_2)$ where we use $p_1, p_2, p_1 \cdot p_2$ as one-point sets. Note that the triples mentioned form the *graph* of the binary operation. Now for purposes of our interpretation $(X_1, X_2, X_3) \in A$ provided there exists $p_1, p_2, p_3 \in M$ such that $p_i \in X_i$, $i=1, 2, 3$ and $p_3 = p_1 \cdot p_2$. Similarly let B be the minimal ternary association in M_1 such that B contains all triples of the form $(q_1, q_2, q_1 \circ q_2)$ for all $q_1, q_2 \in M_1$.

2. Theorem. *With A and B as defined above, t: $M \to M_1$ is a homomorphism if and only if t is (A, B)-continuous.*

Proof. First suppose t is a homomorphism and $(X_1, X_2, X_3) \in A$ then there is $(p_1, p_2, p_3) \in A$ where $p_3 = p_1 \cdot p_2$ and $p_i \in X_i$, $i=1, 2, 3$. Then $(tp_1, tp_2, tp_3) = (tp_1, tp_2, tp_1 \circ tp_2) \in B$ and hence $tA \subseteqq B$ if t is a homomorphism.

Next, suppose t is (A, B)-continuous. Then $(p_1, p_2, p_1 \cdot p_2) \in A$ for each $p_1, p_2 \in M$ and hence $(tp_1, tp_2, t(p_1 \cdot p_2)) \in B$. But tp_1, tp_2, and $t(p_1 \cdot p_2)$ are elements in M_1 and hence necessarily $t(p_1 \cdot p_2) = tp_1 \circ tp_2$ by definition of B. Hence t is a homomorphism. Q.E.D.

The point of this theorem is not so much that the result is obtained as it is that we have now demonstrated a common framework including all the form preserving maps called homomorphisms in mathematics. Why should the simple binary relation-preserving map require triples, whereas topology with its infinities requires ordered pairs? It is because we assume that the binary operation is not necessarily *commutative* that we must use triples. If the binary operations $\cdot$ and $\circ$ had been commutative, then we could have defined A as the binary association generated by all pairs $(\{p_1, p_2\}, p_1 \cdot p_2)$ and B correspondingly. However, it is more convenient in general to use the triples. In order to preserve several functions in one mapping we may put all their graphs in ordered ennuples of sets. Thus if $\cdot$ and $+$ are two binary operations, we may generate an association by using quadruples $(p_1, p_2, p_1 + p_2, p_1 \cdot p_2)$ as a base: we may include ordinary topological continuity by considering, say, $(p_1, p_2, p_1 + p_2, p_1 \cdot p_2, X, p)$ as a base for an association where $p \in uX$ for u a topological closure function.

Suppose M is the set of real numbers and $p_1, p_2, \cdots, p_n, \cdots$ is a convergent sequence. Then we may use $(p_0, p_1, \cdots, p_n, \cdots)$ as a base for an association where p_0 is either one of the p_i for $i \geqq 1$ or p_0 is a limit point of $\{p_n\}$. However, this is equivalent to basing a binary association on (p_0, S) where S is the set of numbers in a convergent sequence and $p_0 \in uS$ where u is the closure function. Hence, the topological definitions of continuity require only ordered *pairs* of sets. Here $\{(p_0, S)\}$ may be considered as the *graph* of the real number topology.

Association preserving maps, however, cover a much wider range of applications than is indicated by using graphs or multiple graphs. We may generate associations from any relations among sets involving a fixed number of sets (finite or infinite). Thus for the real numbers we might define $(X_1, X_2) \in A$ provided $uX_1 \cap uX_2 \neq N$ where u is the closure function. Mappings of the real numbers into themselves which are (A, A)-continuous are of interest to study. One property they have is that every set X dense in a connected subset of the real numbers maps into a set dense in a connected subset.

III. *Properties of Continuous Transformations*

This new form of the definition of continuity is so recent that we have not begun the necessary classifications of separations or their dual associations. However, one important property of continuous mappings is mentioned as stated in the following theorem.

3. THEOREM. *Let A, B, C respectively be ennary associations for spaces M, M_1, and M_2. Then if $t_1: M \to M_1$ is (A, B)-continuous and $t_2: M_1 \to M_2$ is (B, C)-continuous, the composite transformation $t_2t_1: M \to M_2$ is (A, C)-continuous. In particular, if $t: M \to M$ is an (A, A)-continuous mapping then $t^k: M \to M$ is (A, A)-continuous for each integer $k \geqq 2$.*

Proof. We have $t_1A \subseteqq B$, $t_2B \subseteqq C$ whence $t_2(t_1A) \subseteqq t_2B \subseteqq C$ and t_2t_1 is (A, C)-continuous. Note that t_2 is inclusion-preserving and hence $t_1A \subseteqq B$ implies $t_2t_1A \subseteqq t_2B$. Q.E.D.

It will be noted that the proof consumes very little space although the Theorem applies for all the usual homomorphisms, algebraic or topological. Among the problems of classification which we mentioned are those related to descriptions of minimal bases for associations. When, for example, can continuity *at a point* be defined so that a transformation continuous at each point is continuous? We have shown that certain binary separations are characterized by means of the Wallace functions which, in appropriate circumstances, become topological closure functions. There are now the problems of extending this work to functions of several set-variables. Concepts of connectedness of sets depending on triples of sets and so on are now also seen to be useful to consider. We have completely characterized connectedness-preserving transformations as a form of association-preserving continuity for topological spaces.

Conclusion. The definition of continuity we have given here has the satisfactory feeling of being more intuitive than the topological definition—a continuous mapping separates no associated sets. That the definition applies to finite spaces is quite clear and very important. Thus we may say that two people are not separated provided they have at least one grandparent in common. Then it is feasible and sensible to speak of continuity in people-to-people transformations.

The insight into the notion of continuity arising from our definition should not be considered ultimate. We are developing the concepts which go with uniform continuity, Lipschitz conditions, etc., which have a different flavor. Moreover, as we have suggested, there seems no necessity for restricting the transformations involved to those induced by element-to-element mappings and this restriction will have to be dropped to characterize closedness-preserving transformations. An ordinary continuous transformation has the property that the inverse image of closed sets is closed. Now the inverse of a transformation is normally *set-valued* and it is separation-preserving when the transformation is association-preserving. Perhaps we shall want to consider separation-preserving maps under some other heading than continuity, but it is clear that they are dual to association-preserving ones.

The papers most relevant to an understanding of this aspect of extended topology at this time are [1], [2], [3]. These contain references to other works. Reports containing these and the yet unpublished [3] are available at the Numerical Analysis Department, University of Wisconsin.

Finally, it now seems possible to extend the Erlanger Program of Felix Klein to many more systems than the geometries he discussed. Invariance and continuity are essentially dual concepts and we have taken a step here in the direction of demonstrating it.

References

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A TEACHING NOTE

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A standard method for solving a trigonometrical equation of the form $a \sin x + b \cos x = c$ is to put $\tan \frac{1}{2}x = t$.

Applying this to the equation

$$3 \sin x - \cos x = 1$$

we have

$$\frac{6t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1$$

$$6t - 1 + t^2 = 1 + t^2$$

$$6t = 2$$

$$\tan \tfrac{1}{2}x = \tfrac{1}{3}$$

and we fail to find the solutions

$$x = (2n+1)\pi.$$

Of six text books that I have consulted at random, none draws attention to this pitfall.

For rigour, we should begin with some such statement as

"if x is not an odd multiple of π we can put $\tan \frac{1}{2}x = t$."

and end

"It still remains to consider values $x = (2n+1)\pi$."

ON THE ACCURACY OF *Z* CHARTS

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In the familiar Z-type of nomograph illustrated in Figure 1, when x and y are fixed, z is given as $x/(x+ay)$, where

$$a = \frac{\text{length of a unit on the } y \text{ scale}}{\text{length of a unit on the } x \text{ scale}} .$$

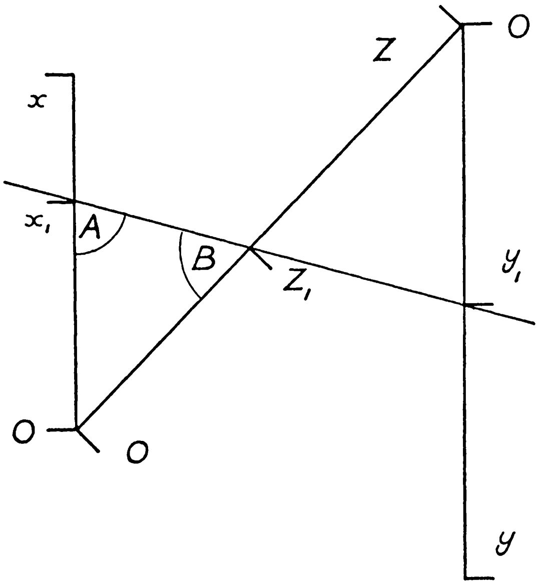

FIG. 1

The x and y scales are parallel, and the z scale may be taken in any direction and of any length between them. Any functions of x, y and z separately may be plotted on the corresponding scales to give the relation

$$f_3(z) = \frac{f_1(x)}{f_1(x) + f_2(y)},$$

but this does not affect our considerations here. We wish to find the optimum position and length of the z scale.

First we shall prove a simple theorem in geometry. It is that the angle subtended at the circumference of a circle by a part-radius extending from the centre is a maximum when the triangle so formed is right-angled at the end of the part-radius away from the centre. With the notation of Figure 2, by the sine rule,

$$\frac{\sin \alpha}{\sin \beta} = \frac{r}{s} .$$

Now s and r are fixed, so that $\sin \beta$ has its maximum when $\sin \alpha$ is a maximum,

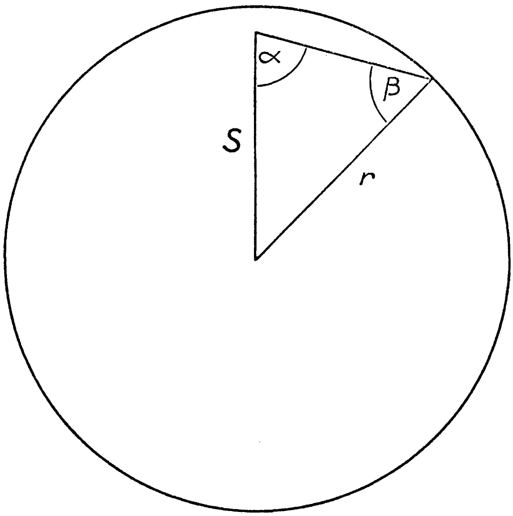

FIG. 2

that is when α is a right angle. Since β is always an acute angle, this also gives the maximum value of the angle β, which proves the theorem.

Returning to the Z chart, let A be the angle between the x scale and the transversal, let B be the angle between the z scale and the transversal, and let b be equal to

$$\frac{\text{length of a unit on the } z \text{ scale}}{\text{length of a unit on the } x \text{ scale}} .$$

Then for particular values of x and y, x_1 and y_1 say, giving $z=z_1$, the triangle formed by the common origin of the x and z scales, and the points $x=x_1$ and $z=z_1$ corresponds to the triangle of the theorem, with angles A and B corresponding to angles α and β in Figure 2.

Now maximum accuracy will be attained in reading the z scale when the angle between it and the transversal is nearest to a right angle. By the construction this angle can not be greater than a right angle, so by the theorem, the maximum accuracy is attained when the transversal is at right angles to the x-scale, and hence also to the y scale.

This principle can be used in the drawing of Z charts, by placing on the same level those portions of the x and y scales most likely to be used together. It will be seen that the length of the z scale, which is determined by the distance between the x and y scales once the position of maximum accuracy is attained, does not affect the accuracy of the reading. This is because only the component of its length parallel to the x and y scales is significant, and this is determined by the position of maximum accuracy. What this means is shown in Figure 3.

The principle would come to its best use in the construction of a nomograph from jointed rods. In this the transversal would be fixed perpendicular to the x and y scales, thus maintaining them in parallel position, while it would remain

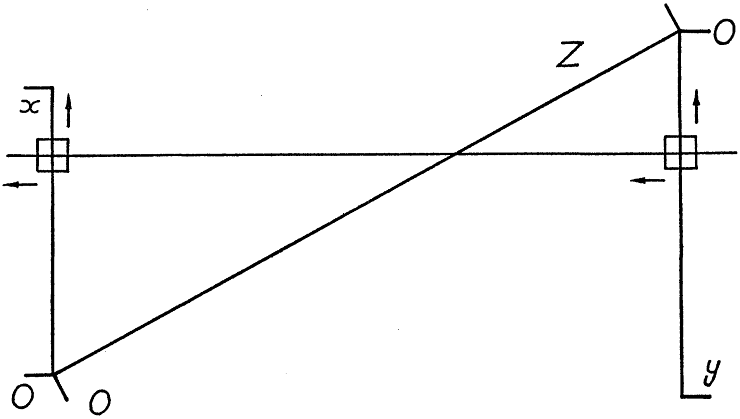

FIG. 3

free to slide across and along the x and y scales. (Imagine two slide rules which have been fixed at right angles by gluing together the upper faces of their cursors.) There would be free joints between the ends of the z scale and the other two scales. So that the transversal could be set for any x and y, it would be necessary for the z scale to be longer than the total length of the other two scales. Beyond this, increase in its length would not affect the accuracy. With such a set of rods the scales would automatically be in the best position for accurate reading in every case.

ANSWERS

A 310. Let $\cos\theta = x$. Then

$$\sqrt{1-x^2} > \frac{1-x}{1+x} \quad \text{or} \quad (1+x)^{3/2} > (1-x)^{1/2}$$

which is obviously true.

A 311. With a convenient radius r and center O describe a circle cutting the line at P and again at A. Draw AO extended to meet the circle again at B. BP is perpendicular to the line at P since angle BPA is a right angle, being inscribed in a semi-circle. This method requires one compass opening and the drawing of one circle and two lines, as opposed to two compass openings, three circles and one line by the more conventional method. If P is at the end of a line segment an additional line would have to be drawn to use the conventional method.

A 312. Orthogonally project the ellipse into a circle. The equilateral inscribed or circumscribed triangles will become inscribed or circumscribed non-equilateral triangles whose centroids cannot coincide with the center of the circle. Since centroids transform into centroids, the proof is completed.

(Quickies on page 142)

VERTEX POINTS OF CURVES

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In the plane, the points of inflection of a curve are independent of the coordinate system. But there are other points on the curve which may be of some interest, for example, the vertex of a parabola that in general cannot be obtained by the technique of taking derivatives. In this article we would like to explain a method of obtaining vertex points of curves in the plane.

1. Tangent conic to a curve. Let $f(x, y)$, a real function of real variables x and y, be of class C'' at a point (a, b), and at least one of the first and second partial derivatives of f is different from zero. Then

$$(1.1)\qquad (x-a \quad y-b \quad 1)\begin{pmatrix} \dfrac{\partial^2 f}{\partial a^2} & \dfrac{\partial^2 f}{\partial a\partial b} & \dfrac{\partial f}{\partial a} \\ \dfrac{\partial^2 f}{\partial a\partial b} & \dfrac{\partial^2 f}{\partial b^2} & \dfrac{\partial f}{\partial b} \\ \dfrac{\partial f}{\partial a} & \dfrac{\partial f}{\partial b} & 0 \end{pmatrix}\begin{pmatrix} x-a \\ y-b \\ 1 \end{pmatrix} = 0,$$

where for example $\partial^2 f/\partial a\partial b$ means the value of $\partial^2 f/\partial x\partial y$ at (a, b), is called the tangent conic of $f(x, y)$ at (a, b). If we expand (1.1), we get the Taylor expansion of $f(x, y)$ at (a, b) up to the second degree terms. If at (a, b) the second derivatives are all zero and at least one of the first partial derivatives is not zero, then (1.1) becomes the tangent plane at (a, b). We shall call

$$Q = \begin{pmatrix} \dfrac{\partial^2 f}{\partial a^2} & \dfrac{\partial^2 f}{\partial a\partial b} \\ \dfrac{\partial^2 f}{\partial a\partial b} & \dfrac{\partial^2 f}{\partial b^2} \end{pmatrix}$$

the Hessian matrix of $f(x, y)$. We denote the matrix of (1.1) by A.

2. Projection on the tangent and normal. The formula

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

shows that

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

is the direction of the normal to $f(x, y) = 0$ at a point (x, y). Let the origin be at the point at which the normal is

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Then

$$\alpha = \left(\frac{\dfrac{\partial f}{\partial x}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 = \left(\dfrac{\partial f}{\partial y}\right)^2}}, \frac{\dfrac{\partial f}{\partial y}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}} \right)$$

is a unit vector on the normal. Thus the projection of any vector $\xi = (X, Y)$ on the normal is $(\xi, \alpha)\alpha$ [Fig. 1]. Here (ξ, α) is the inner product of the vectors ξ and α. Therefore the equations of this projection will be obtained from

$$\left[X \frac{\dfrac{\partial f}{\partial x}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}} + Y \frac{\dfrac{\partial f}{\partial y}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}} \right] \alpha.$$

Thus

$$\begin{cases} X_1 = X \dfrac{\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2} + Y \dfrac{\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial f}{\partial y}\right)}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2} \\ Y_1 = X \dfrac{\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial f}{\partial y}\right)}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2} + Y \dfrac{\left(\dfrac{\partial f}{\partial y}\right)^2}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}. \end{cases}$$

Thus the matrix of this projection is:

$$P = \begin{pmatrix} \dfrac{\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2} & \dfrac{\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial f}{\partial v}\right)}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2} \\ \dfrac{\left(\dfrac{\partial f}{\partial x}\right)\left(\dfrac{\partial f}{\partial y}\right)}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)} & \dfrac{\left(\dfrac{\partial f}{\partial y}\right)^2}{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2} \end{pmatrix},$$

and the matrix of the projection on the tangent line will be $I - P$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Conic curvature. If (1.1) is a perfect square at (a, b), then $f(x, y)$ is called doubly flat at this point. Suppose (1.1) is not a perfect square at (a, b).

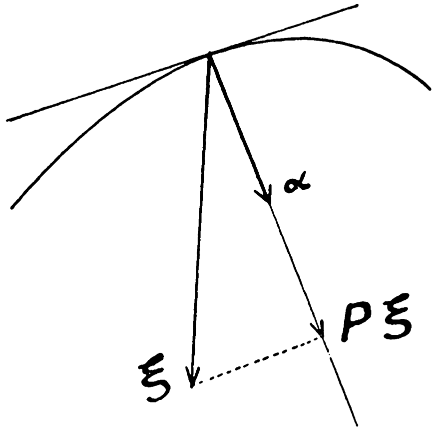

FIG. 1

Then by (89), page 85 of [1] centers of the conic (1.1) may be obtained by

$$(3.1)\qquad ([x-a]\quad [y-b])\begin{bmatrix} \dfrac{\partial^2 f}{\partial a^2} & \dfrac{\partial^2 f}{\partial a\partial b} \\ \dfrac{\partial^2 f}{\partial a\partial b} & \dfrac{\partial^2 f}{\partial b^2} \end{bmatrix} = \left(-\frac{\partial f}{\partial a}\ -\frac{\partial f}{\partial b}\right).$$

This is a system of two equations and two unknowns. The following cases may occur

$$\text{I.}\qquad \begin{vmatrix} \dfrac{\partial^2 f}{\partial a^2} & \dfrac{\partial^2 f}{\partial a\partial b} \\ \dfrac{\partial^2 f}{\partial a\partial b} & \dfrac{\partial^2 f}{\partial b^2} \end{vmatrix} \neq 0.$$

Then there is a unique center which is called the center of conic curvature of $f(x, y)$ at (a, b) [Fig. 2].

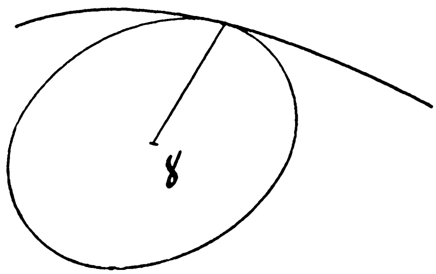

FIG. 2

II. Let the determinant in I be zero, and

$$(3.2)\qquad \frac{\partial^2 f}{\partial a^2}\Big/\frac{\partial^2 f}{\partial a\partial b} = \frac{\partial f}{\partial a}\Big/\frac{\partial f}{\partial b}\,.$$

Then there are many centers for the conic (1.1), namely, all the points of the line

$$(x-a)\frac{\partial^2 f}{\partial a^2}+(y-b)\frac{\partial^2 f}{\partial a\partial b}=-\frac{\partial f}{\partial a}. \tag{3.3}$$

The equalities (3.2) imply that

$$\frac{\partial^2 f}{\partial a^2}=k\frac{\partial f}{\partial a},\qquad \frac{\partial^2 f}{\partial a\partial b}=k\frac{\partial f}{\partial b}, \tag{3.4}$$

where k is a real number. Thus (3.3) can be written as

$$\frac{\partial f}{\partial a}(x-a)+\frac{\partial f}{\partial b}(y-b)=-\frac{1}{k}\frac{\partial f}{\partial a}.$$

We observe that this line is parallel to the tangent line of the curve. We will choose a point on this line which is at the shortest distance from (a, b). This can be done by getting the intersection of (3.3) and the line

$$(x-a)\frac{\partial^2 f}{\partial a\partial b}-(y-b)\frac{\partial^2 f}{\partial a^2}=0,\quad\text{or}\quad \frac{\partial f}{\partial b}(x-a)-\frac{\partial f}{\partial a}(y-b)=0 \tag{3.5}$$

which is perpendicular to (3.3) and passes through (a, b). Thus the system of equations consisting of (3.3) and (3.5) gives the center which is

$$x=a-\frac{\dfrac{\partial^2 f}{\partial a^2}\,\dfrac{\partial f}{\partial a}}{\left(\dfrac{\partial^2 f}{\partial a^2}\right)^2+\left(\dfrac{\partial^2 f}{\partial a\partial b}\right)^2},\qquad y=b-\frac{\dfrac{\partial^2 f}{\partial a\partial b}\,\dfrac{\partial f}{\partial a}}{\left(\dfrac{\partial^2 f}{\partial a^2}\right)^2+\left(\dfrac{\partial^2 f}{\partial a\partial b}\right)^2}.$$

III. Let the determinant in I be zero, but

$$\frac{\partial^2 f}{\partial a^2}\Big/\frac{\partial^2 f}{\partial a\partial b}\neq\frac{\partial f}{\partial a}\Big/\frac{\partial f}{\partial b}.$$

Then (1.1) does not have a center. In this case the equation (3.1) gives a set of

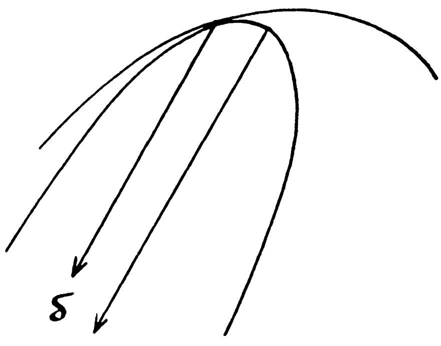

FIG. 3

parallel lines. We call the direction of these lines the direction of conic curvature of $f(x, y)$ at (a, b) [Fig. 3].

4. Example. Find the center of conic curvature of

$$y - x^2 \sin y + 3x - 9$$

at $p=(3, 0)$.

Solution:

$$\frac{\partial f}{\partial x} = -2x \sin y + 3, \quad \text{at } P, \quad \frac{\partial f}{\partial x} = 3;$$

$$\frac{\partial f}{\partial y} = 1 - x^2 \cos y, \quad \text{at } P, \quad \frac{\partial f}{\partial y} = -8;$$

$$\frac{\partial^2 f}{\partial x^2} = -2 \sin y, \quad \text{at } P, \quad \frac{\partial^2 f}{\partial x^2} = 0;$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 \sin y, \quad \text{at } P, \quad \frac{\partial^2 f}{\partial y^2} = 0;$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2x \cos y, \quad \text{at } P, \quad \frac{\partial^2 f}{\partial x \partial y} = -6.$$

Therefore the center is the solution of

$$([x-3] \quad y)\begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} = (-3 \quad 8)$$

which gives $(7/2, -4/3)$. We leave it to the reader to supply examples for other cases.

5. Vertex points. A vertex point of a curve is a point at which the point of contact of tangent conic with the curve is a vertex of the conic. It is clear that at a vertex point the projection of the center or direction of curvature on the tangent line is zero [Fig. 4].

THEOREM. *A necessary and sufficient condition for a point to be a vertex point of a curve is that at that point*

$$PQ = QP,$$

where P and Q are the matrices described in sections 1 *and* 2.

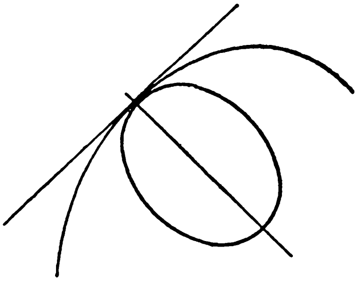

FIG. 4

Proof. By section 2 the center or the direction of conic curvature of $f(x, y)$ is obtained from

$$(x - a \quad y - b)\begin{pmatrix} \dfrac{\partial^2 f}{\partial a^2} & \dfrac{\partial^2 f}{\partial a \partial b} \\ \dfrac{\partial^2 f}{\partial a \partial b} & \dfrac{\partial^2 f}{\partial b^2} \end{pmatrix} = \left(-\frac{\partial f}{\partial a} \quad -\frac{\partial f}{\partial b}\right).$$

I. If the matrix Q is non-singular, than Q^{-1}, i.e., the inverse of Q exists. Let

$$Q^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then

$$(x - a \quad y - b) = \left(-\frac{\partial f}{\partial a} \quad -\frac{\partial f}{\partial b}\right)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The projection of the vector $(x-a, y-b)$ on the tangent line is zero, i.e.,

$$(x - a \quad v - b) \quad (I - P) = 0.$$

Thus

$$\left(-\frac{\partial f}{\partial a} \quad -\frac{\partial f}{\partial b}\right) Q^{-1}(I - P) = 0.$$

This implies that

$$\left(\frac{\partial f}{\partial a} \quad \frac{\partial f}{\partial b}\right) Q^{-1} = \left(\frac{\partial f}{\partial a} \quad \frac{\partial f}{\partial b}\right) Q^{-1} P,$$

or

$$\left(\frac{\partial f}{\partial a} \quad \frac{\partial f}{\partial b}\right) = \left(\frac{\partial f}{\partial a} \quad \frac{\partial f}{\partial b}\right) Q^{-1} P Q.$$

Therefore

$$Q^{-1}PQ = P \quad \text{or} \quad PQ = QP.$$

II. Suppose Q is singular, and the equality (3.2) holds. Then the center of conic curvature of f is already on the normal line. Thus the point (a, b) is a vertex point of the curve. Suppose $\partial f/\partial y \neq 0$ at (a, b). Then (3.2) implies

$$\frac{\dfrac{\partial^2 f}{\partial x^2}}{\dfrac{\partial^2 f}{\partial x \partial y}} = \frac{\dfrac{\partial^2 f}{\partial x \partial y}}{\dfrac{\partial^2 f}{\partial y^2}} = \frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = k.$$

Substituting these in Q and P we get

$$Q = \frac{\partial^2 f}{\partial y^2}\begin{pmatrix} k^2 & k \\ k & 1 \end{pmatrix},$$

and

$$P = \frac{1}{(k^2+1)}\begin{pmatrix} k^2 & k \\ k & 1 \end{pmatrix}.$$

Now it is clear that $PQ = QP$. Other cases such as $\partial f/\partial y = 0$, etc. should be discussed.

III. Suppose Q is singular and the curve does not have a center of conic curvature. Then

$$\frac{\partial^2 f}{\partial a^2}(x-a) + \frac{\partial^2 f}{\partial a \partial b}(y-b) = 0$$

is parallel to the direction of conic curvature. Thus

$$\left(\frac{\partial^2 f}{\partial a \partial b},\ -\frac{\partial^2 f}{\partial a^2}\right)$$

is the direction of conic curvature of the curve. On the other hand

$$\left(\frac{\partial^2 f}{\partial b^2},\ -\frac{\partial^2 f}{\partial a \partial b}\right)$$

is also the direction of conic curvature of the curve. At a vertex point these directions must be the same as

$$\left(\frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}\right).$$

Thus,

$$\text{(5.1)} \qquad \frac{\dfrac{\partial^2 f}{\partial a \partial b}}{-\dfrac{\partial^2 f}{\partial a^2}} = \frac{\dfrac{\partial^2 f}{\partial b^2}}{-\dfrac{\partial^2 f}{\partial a \partial b}} = \frac{\dfrac{\partial f}{\partial a}}{\dfrac{\partial f}{\partial b}} = m,$$

where m is a real number.

Thus

$$Q = \frac{\partial^2 f}{\partial a^2}\begin{pmatrix} 1 & -m \\ -m & m^2 \end{pmatrix},$$

and

$$P = \frac{1}{m^2+1}\begin{pmatrix} m^2 & m \\ m & 1 \end{pmatrix}.$$

Now it is easy to see that in this case

$$PQ = QP = 0.$$

If $\partial f/\partial b = 0$, then we choose reciprocals of the ratios in (5.1).

6. Example. Find the vertex points of

$$\frac{\sqrt{2}}{2}(x-y) - \sin\frac{\sqrt{2}}{2}(x-y) = 0.$$

Note that this is the curve $y=\sin x$ which has been rotated through an angle 45°. The reader may compute P and Q and set $PQ=QP$. This gives

$$\sin\frac{\sqrt{2}}{2}(x+y)\cos\frac{\sqrt{2}}{2}(x+y) = 0.$$

We observe that the points satisfying this equation are the ones obtained by rotating maximum, minimum, and inflection points of $y=\sin x$ through an angle 45°.

7. Curves of fixed center of conic curvature. Let Q be non-singular. Then the equation (3.1) gives the center of conic curvature. We would like to show that a curve with a fixed center of conic curvature is not necessarily a conic section. For convenience let the center be (0, 0) and the point (a, b) be on point (x, y) of the curve. Then the equation (3.1) will be

$$\text{(7.1)}\qquad \begin{cases} x\dfrac{\partial^2 f}{\partial x^2} + y\dfrac{\partial^2 f}{\partial x\partial y} = \dfrac{\partial f}{\partial x} \\[2ex] x\dfrac{\partial^2 f}{\partial x\partial y} + y\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial f}{\partial y}\cdot \end{cases}$$

We shall solve these equations as follows:

Let $\partial f/\partial x = \phi(r, \theta)$, $\partial f/\partial y = \psi(r, \theta)$. Then

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial \phi}{\partial r}\cos\theta - \frac{\partial \phi}{\partial \theta}\frac{\sin\theta}{r},$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial \phi}{\partial r}\sin\theta + \frac{\partial \phi}{\partial \theta}\frac{\cos\theta}{r},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial \psi}{\partial r}\sin\theta + \frac{\partial \psi}{\partial \theta}\frac{\cos\theta}{r},$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial \psi}{\partial r}\cos\theta - \frac{\partial \psi}{\partial \theta}\frac{\sin\theta}{r}\cdot$$

Then (7.1) becomes

$$\begin{cases} r\dfrac{\partial\phi}{\partial r} = \phi \\ r\dfrac{\partial\psi}{\partial r} = \psi. \end{cases}$$

Thus $\phi = rA(\theta)$ and $\psi = rB(\theta)$. Let $f = U(r, \theta)$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial U}{\partial r}\cos\theta - \frac{\partial U}{\partial\theta}\frac{\sin\theta}{r} = rA(\theta),$$

$$\frac{\partial f}{\partial y} = \frac{\partial U}{\partial r}\sin\theta + \frac{\partial U}{\partial\theta}\frac{\cos\theta}{r} = rB(\theta).$$

Therefore we get

$$\frac{\partial U}{\partial r} = r[A\cos\theta + b\sin\theta] \tag{7.2}$$

$$\frac{1}{r}\frac{\partial U}{\partial\theta} = r[-A\sin\theta + B\cos\theta]. \tag{7.3}$$

Let $P(\theta) = A\cos\theta + B\sin\theta$ and $Q(\theta) = -A\sin\theta + B\cos\theta$. Then from (7.3) we get

$$U(r, \theta) = r^2\int Q(\theta)d\theta + C(r).$$

Comparing this with (7.2), we get

$$2r\int Q(\theta)d\theta + C'(r) = rP(\theta).$$

Choose $C'(r) = 2kr$, k constant. Thus $C(r) = kr^2 + h$. Therefore

$$\int Q(\theta)d\theta = \frac{P(\theta) - 2k}{2},$$

and the solution of (7.1) will be

$$\begin{cases} U = \dfrac{r^2}{2}P(\theta) + h \\ 2\displaystyle\int Q(\theta)d\theta = P(\theta) - 2k. \end{cases} \tag{7.4}$$

To obtain a solution, we observe that the second equation of (7.4) is the same as

$$A'\cos\theta + A\sin\theta = B\cos\theta - B'\sin\theta.$$

Thus

$$A \sec\theta = \int (B\sec\theta - B'\sec\theta\tan\theta)d\theta + C.$$

Choose $B=1$, and $C=0$. Then we get

$$U = \frac{r^2}{2}\cos^2\theta\log(\sec\theta+\tan\theta) + \frac{r^2}{2}\sin\theta.$$

Thus

$$f(x,y) = \frac{x^2}{2}\log\left(\frac{\sqrt{x^2-y^2}}{x}+\frac{y}{x}\right)+\frac{y}{2}\sqrt{x^2+y^2},$$

which is not a conic section.

8. Conjecture. Vertex points of a curve are points of maximum or minimum curvature.

The reader may study this conjecture and prove that it is a theorem.

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KAPPA AND TAU

Divide into derivative secundus
The one-half power of the power three
Of primus squared to which is added one. Thus
We measure curvedness rectilinearly.

Perhaps—indeed, this is a little boring,
This dull receipt, this quantitative scoring
Of small amounts we turn and twist
In flatland where we all subsist.
How hard it is to twist from this and, soaring
On wings of torsion, coexist.

MARLOW SHOLANDER

AN ABSTRACT-ALGEBRAIC PROOF OF THE CHINESE REMAINDER THEOREM

LEO UNGER, Litton Systems, Inc.

The Chinese remainder theorem states that a system of linear congruences

$$\{x \equiv r_i(m_i)\} \qquad i = 1, 2, \cdots n \tag{1}$$

with pairwise relatively prime moduli m_i has a unique solution modulo the product of the moduli $\prod_{i=1}^{n} m_i = M$.

Proof. Multiplying every equation of (1) by a corresponding ${}^iM = M/m_i$ and adding the results yields

$$x\left(\sum_{i=1}^{n} {}^iM\right) = \sum_{i=1}^{n} r_i\, {}^iM(M). \tag{2}$$

Since by hypothesis $(m_r, m_s) = 1$ for $r \neq s$ it follows that

$$\left(M, \sum_{i=1}^{n} {}^iM\right) = 1. \tag{3}$$

Therefore there exist integers A and B such that

$$AM + B\left(\sum_{i=1}^{n} {}^iM\right) = 1$$

or

$$B \equiv \left(\sum_{i=1}^{n} {}^iM\right)^{-1} (M). \tag{4}$$

Multiplying (2) by (4) yields the solution

$$x \equiv \left(\sum_{i=1}^{n} r_i\, {}^iM\right)\left(\sum_{i=1}^{n} {}^iM\right)^{-1} (M). \tag{5}$$

The uniqueness of the solution follows from the uniqueness of the inverse modulo M of (4). For if y were another solution it would have to satisfy (2) so that

$$y\left(\sum_{i=1}^{n} {}^iM\right) \equiv \sum_{i=1}^{n} r_i\, {}^iM(M).$$

Multiplying this by B found before yields an expression for y identical to (5) for x, so $y \equiv x(M)$.

The bulk of the actual solution involves the determination of the multiplicative inverse B of $\sum {}^iM$ in J_M, the ring of integers modulo M. Such an inverse necessarily exists since by (3) $\sum {}^iM$ is not a zero divisor in J_M. Two methods for finding B suggest themselves. One is the construction of a multiplication table for J_M in which B could be looked up. This may be laborious if M is large. Another way is the performance of the Euclidean algorithm on $\sum^i M$ and M in which the last non-vanishing remainder will be the g.c.d. of these numbers. By (3) this g.c.d. is 1. Then, expressing the successive remainders as linear combinations of M and $\sum^i M$ will yield B.

THE NATURE OF $N = n(n+1) - 1$

CHARLES W. TRIGG, Los Angeles City College

The only composite value of N for $n \leqq 11$ is $7\cdot 8-1=5\cdot 11$. However, if $N=n(n+1)-1$ is divisible by p, then $(n+p)(n+p+1)-1$ or $N+p(2n+p+1)$ is divisible by p also. Hence, the density of prime values of N tends to decrease for large n. The frequency of primes and of composite numbers in the first ten decades of N for $n \leqq 100$ are:

n	p	pp_2	pp_2p_3	pp_2^2	p^3
1–10	9	1	—	—	—
11–20	6	4	—	—	—
21–30	5	5	—	—	—
31–40	4	5	—	—	1
41–50	6	2	1	1	—
51–60	6	3	1	—	—
61–70	5	4	1	—	—
71–80	1	8	1	—	—
81–90	4	5	—	1	—
91–100	4	5	1	—	—
1–100	50	42	5	2	1

The smallest value of N which is a cube is $36\cdot 37-1=11^3$. Two values of n, 42 and 84, yield N's which have a square factor. The values of n for which N is the product of three distinct primes are 47, 52, 62, 80 and 92.

There are seven palindromic values of N: 11, 55, 181, 505, 929, 1331, and 9119.

The terminal digits of consecutive values of N repeat the symmetrical cycle 9, 1, 5, 1, 9.

Neither $(3k-1)(3k)-1$, $3k(3k+1)-1$, nor $(3k+1)(3k+2)-1=9k(k+1)+1$ is divisible by 3. Hence, no value of N can have 3 as a factor. In like manner, it may be shown that 2, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, and 97 cannot be factors of N. In fact, this follows since none of these appear as factors in the first 100 values of N.

$$(2 + 5k)(3 + 5k) = 5m + 2\cdot 3 - 1,$$

so every fifth value of N beginning with $2\cdot 3-1=5$ contains 5 as a factor. Similarly, it may be shown that each prime regularly appears as a factor in every pth position after its first appearance. 11 as a factor appears regularly in two cycles, one starting with $3\cdot 4-1$, the other with $7\cdot 8-1$.

TEACHING OF MATHEMATICS

EDITED BY ROTHWELL STEPHENS, Knox College

This department is devoted to the teaching of mathematics. Thus, articles of methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as a teacher, are interested, or questions which you would like others to discuss, should be sent to Rothwell Stephens, Mathematics Department, Knox College, Galesburg, Illinois.

RESEARCH AND THE MATHEMATICIAN

WILLIAM EDWARD CHRISTILLES, St. Mary's University, Texas

Due to the constantly increasing expansion of knowledge, the subsequent demand upon the university professor to stay abreast of the current developments in his field and his further obligation of producing and publishing scholarly research, there has resulted a noticeable decline upon the part of the professor in aiding the students of mathematics to do independent and creative investigations.

One major reason for this deficit is the unrealistic demands placed upon many of our professors in the way of overburdened teaching loads, unnecessary and time consuming administrative jobs, and numerous other tasks which take the professor away from his major duty, that of working with his students. To be able to adequately supervise research, even on the undergraduate or beginning graduate level, the professor must devote a certain amount of his own time to research. This will enable him to keep well abreast of his student's progress and further allow him time to properly advise them.

The fact that many colleges and universities are constantly pushing the beginning of creative and scholarly work in mathematics to a later period in the student's curriculum is evidence enough of a dangerous trend. Also, it does not seem that dropping the writing of a thesis from the masters program in mathematics, currently in practice in many of our institutions of higher learning, and replacing it with more advanced course work is a particularly healthy step forward. An even greater danger is the proposed Doctor of Arts degree in mathematics. This degree would replace the creative and scholarly thesis with an expository paper which would ultimately result in the downgrading of the doctorate in a similar manner to that already suffered by the masters degree. Certainly the need of our students to obtain both a greater depth of understanding and a broader scope in mathematics is apparent; yet, this need should not be allowed to overlap into the student's natural process of development. Further care must be taken not to allow the trend of prolonging the educational development of the student to continue to a point where it will reach into the most productive years of his life.

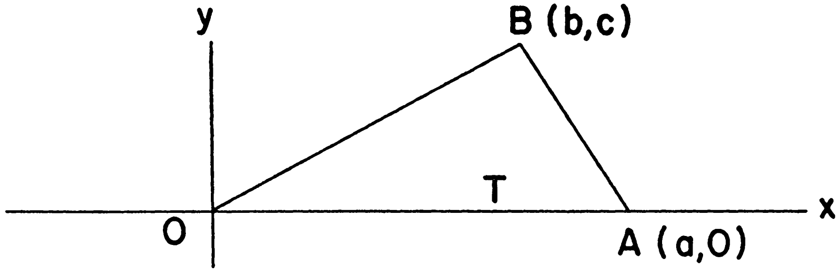
Finally, it seems that this failure to educate our students as productive mathematicians is principally due to an increased emphasis upon rating a mathematician by the degree he possesses rather than by his performance. If the student is aware he is to be judged more by the degrees he obtains than by his accomplishments, he will begin to concentrate his efforts toward that end. Thus the curriculums in many of our colleges and universities need to be coordinated in a more realistic manner, giving adequate attention to student research. More practical and exacting means for evaluating the students must be developed. Only then will our colleges and universities be able to educate a sufficient number of mathematicians having the necessary depth in their field and capable also of using their knowledge in a productive manner.

ANALYTIC PROOF OF THE FEUERBACH THEOREM

J. M. STARK, Lamar State College of Technology

It is the object of this paper to show that by use of geometric and algebraic symmetry, and by judicious choice of notation, one can provide proofs of the Feuerbach Theorem and properties of the Euler line which are at the same time not prohibitively tedious nor more sophisticated than analytic geometry. This is accomplished in the case of the Feuerbach Theorem by showing that a single computation suffices to establish tangency between one circle and four others (Lemma 3 below). We conclude with remarks indicating that the material in this paper is suitable for introducing the beginning college student to the very important idea that sometimes in mathematics simplicity may be purchased by paying a price in motivation.

By triangle T we understand the triangle in the cartesian plane with vertices $O(0, 0)$, $A(a, 0)$, and $B(b, c)$. We suppose $a>0$, $b>0$, $c>0$.



As a matter of notation, let $\beta^2=b^2+c^2$, $\delta^2=(a-b)^2+c^2$, $\lambda=a+\beta+\delta$. Note that $\lambda\neq 0$ regardless of whether β, δ are taken positive or negative. This follows from the fact that the sum of two sides of a proper triangle is greater than the third side.

LEMMA 1. *The equation*

$$[x - a(b + \beta)/\lambda]^2 + (y - ac/\lambda)^2 = (ac/\lambda)^2 \tag{1}$$

represents a circle tangent to the three sides of triangle T.

The lines $\overline{OA}$, $\overline{OB}$, $\overline{AB}$ determined by the sides of T are represented respectively by the equations

$$y = 0, \tag{2}$$

$$v = (c/b)x, \tag{3}$$

(4) $$x = [(b - a)y + ac]/c,$$

as may be verified by observing that these are linear equations satisfied by the coordinates of pairs of vertices of T. Substituting successively (2), (3), (4) into (1) and using the definitions of λ, β, δ, we obtain respectively

(5) $$[x - a(b + \beta)/\lambda]^2 = 0,$$

(6) $$[(\beta/b)x - a(b + \beta)/\lambda]^2 = 0,$$

(7) $$[(\delta/c)y - a(\delta + a - b)/\lambda]^2 = 0,$$

which exhibit equal roots and prove Lemma 1.

LEMMA 2. *The circle N:*

(8) $$[x - (a + 2b)/4]^2 + [y - (ab + c^2 - b^2)/(4c)]^2 = \beta^2\delta^2/(16c^2),$$

where $\beta^2 = b^2 + c^2$, $\delta^2 = (a - b)^2 + c^2$, passes through the midpoints of the sides of triangle T.

Equation (8) may be rewritten as

(9) $$x^2 + y^2 - (1/2)(a + 2b)x - [1/(2c)](c^2 + ab - b^2)y + (1/2)ab = 0.$$

The mid-points of the sides of triangle T have coordinates $(a/2, 0)$, $(b/2, c/2)$, and $[(a+b)/2, c/2]$. Since the coordinates of these points satisfy (9), we have that N is the circle through the mid-points of the sides of T.

Let $p, r, s, \rho, \sigma, \tau$ denote variables satisfying $s \neq 0$, $\rho \neq 0$. Then we define the function $\Omega \equiv \Omega(p, r, s, \rho, \sigma, \tau)$ as follows:

$$\Omega \equiv [p(r + \sigma)/\rho - (p + 2r)/4]^2 + [ps/\rho - (pr + s^2 - r^2)/(4s)]^2 - [\sigma\tau/(4s) - ps/\rho]^2.$$

LEMMA 3. *Let a, b, c, be any three real numbers satisfying $a > 0$, $b > 0$, $c > 0$. Then $\Phi \equiv \Omega(a, b, c, \lambda, \beta, \delta) = 0$ whenever $\lambda = a + \beta + \delta$, $\beta^2 = b^2 + c^2$, $\delta^2 = (a - b)^2 + c^2$.*

To prove Lemma 3, we substitute $a, b, c, \lambda, \beta, \delta$ for $p, r, s, \rho, \sigma, \tau$, respectively, in the definition of Ω, then square the brackets as indicated to obtain

$$\begin{aligned}\Phi = {} & a^2(b + \beta)^2/\lambda^2 - a(b + \beta)(a + 2b/(2\lambda) + (a + 2b)^2/(16) \\ & - a(c^2 + ab - b^2)/(2\lambda) + (c^2 + ab - b^2)^2/(16c^2) - \delta^2\beta^2/(16c^2) \\ & + \delta\beta a/(2\lambda).\end{aligned}$$

In this last expression for Φ, we obtain $ab/2$ as the sum of all terms exhibiting 16 as a factor in the denominator, provided we substitute for β^2 and δ^2 their expressions in terms of a, b, c. In this manner the last expression for Φ yields

$$(2\lambda^2/a)\Phi = 2a(b + \beta)^2 + \lambda[b\lambda - (b + \beta)(a + 2b) - (c^2 + ab - b^2) + \delta\beta].$$

Upon substituting $\lambda = a + \beta + \delta$, the last equation gives

$$\begin{aligned}(2\lambda^2/a)\Phi = {} & \delta[\beta^2 - b^2 - c^2] + \beta[\delta^2 - (a - b)^2 - c^2] + a(\beta^2 + b^2 - c^2) \\ & + b(\delta^2 - \beta^2 - a^2),\end{aligned}$$

the right hand side of which vanishes identically, *regardless of the signs of* β, δ, upon substituting for β^2, δ^2 their expressions in terms of a, b, c.

COROLLARY 4. *Circle* N *is tangent to circle* (1).

In general, tangency of two circles may be established by showing that the square of the distance between their centers is equal to either the square of the sum of their radii or the square of the difference of their radii, depending upon whether the circles are tangent externally or internally. That one of these situations is the case for circles N and (1) is established by the vanishing of $\Phi \equiv \Omega(a, b, c, \lambda, \beta, \delta)$, which is the conclusion of Lemma 3.

The circle through the mid-points of the sides of a triangle is called the nine-point circle of the triangle because of the

NINE-POINT CIRCLE THEOREM. *The circle through the mid-points of the sides of a triangle also passes through the feet of the altitudes and the mid-points of the segments joining the orthocenter to the vertices.*

We have shown that circle N is the circle through the mid-points of the sides of triangle T. The foot of the altitude from vertex B has coordinates $x=b$, $y=0$, which satisfy (9). Since for a given triangle, B is an *arbitrary* vertex, we have that the circle through the mid-points of the sides of a triangle passes through *all* the feet of the altitudes.

The point $H[b, b(a-b)/c]$ is the orthocenter of triangle T. For, $\overline{BH}: x=b$ is clearly an altitude of T, and $\overline{OH}: y=[(a-b)/c]x$ is perpendicular to $\overline{AB}$ as is shown by comparison to (4). The mid-point $[b/2, b(a-b)/(2c)]$ of the segment $\overline{OH}$ is shown to lie on circle N by substituting the coordinates of this point into (9). As above in the case of the feet of the altitudes, it follows that the circle N passes through the mid-points of all three of the segments joining the orthocenter to the vertices.

FEUERBACH THEOREM. *The nine-point circle of a triangle is tangent to the incircle and to the three excircles of the triangle.*

At this point we need only to discuss the four possibilities arising from the choice of signs for β and δ to make it clear that Corollary 4 provides a proof of the Feuerbach Theorem. In all of the above work, we have been very careful to avoid choosing algebraic signs for β and δ. Now observe that equation (1) represents different circles for each of the four choices of algebraic signs for the pair β, δ. These four circles are the incircle and the three excircles of the triangle T. From the definition and properties of the function Ω, it is easily shown that of the four circles just mentioned, only that one with $\beta>0$, $\delta>0$ is tangent internally to the nine-point circle N. This is the situation with the inscribed circle of triangle T. Thus the vanishing identically of the function Ω under the conditions stated is, in a sense, the "heart" of the Feuerbach Theorem.

We mention in passing that using the above propositions, the properties of the Euler line and other theorems pertaining to the geometry of the triangle are readily obtainable.

Remarks. In the above demonstrations, derivations and motivations not required for proofs are avoided almost as much as possible. (E.g., the circle N of Lemma 2 is presented before its properties are established.) It is realized that presenting mathematics in this fashion usually provokes a lively discussion between the instructor and beginning college students.

Many immature students attempt to require of all mathematical proofs that, save for what is hypothesized, all other facts and quantities used in a proof not only must be "derived" as a part of the proof, but must be immediately motivated as well. On the other hand, the instructor realizes that rigorous proofs can be presented and good pedagogy practiced when at the same time motivations are left to take one or several forms in the minds of the individual students. The atmosphere in the classroom is improved when students realize that the instructor can function rigorously and effectively without always supplying motivations or without continuously answering the question: "Where did it come from and what is it good for?"

It has been found that if the student is first subjected to proofs of the Feuerbach Theorem using more sophisticated tools, then the above demonstrations can be used to persuade him that for an expeditious presentation of a certain body of material, it is sometimes better temporarily to sacrifice motivation in favor of using more elementary tools.

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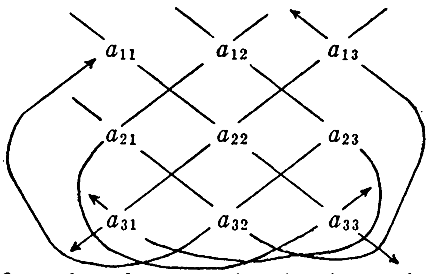
CONCERNING DIAGRAMS FOR DETERMINANTS

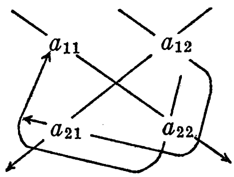
H. W. GOULD, West Virginia University

There have been several provocative discussions of diagrammatic methods for expanding determinants. Since almost everyone likes to join in and warn his students against some unconscious assumption that the usual diagram method works for fourth-order, fifth-order, etc. determinants and so by "induction" for nth order determinants, it may be of interest to point out an unconscious assumption made by many persons in the reverse direction.

The common diagram for third-order determinants



does not work in the first place for second-order determinants. If it did we should have



so that all second-order determinants would have value 0. This would simplify a large part of mathematics. Strangely though, many who realize that the common diagram for third-order determinants does not work for higher order fail to realize that it does not work for second-order determinants.

NON ASSOCIATIVE STRUCTURES*

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The purpose of the present note is to indicate some examples of simpler structures than loops (see [1]) and near-rings (see [2]).

As is usual, we will denote by the same symbol a set and the structure obtained by the consideration of one or more operations among its elements.

Definition 1. Let K be a division ring and * an operation defined by the mapping $K\times K\rightarrow K$ that to every ordered pair $(a, b)\in K\times K$ $(b\neq 0)$ there corresponds the unique element ab^{-1}.

PROPOSITION 1. *The set* $K^*=K-\{0\}$ *is a right loop relative to the operation* *.

Proof. We note first that the identity i in the division ring K is a right unity,

$$a * i = a,$$

although not left unity.

Also the equations

$$a * x = b$$

$$y * a = b$$

are solvable for any $(a, b)\in K^*\times K^*$,

$$x = b^{-1}a, \qquad y = ba.$$

Remark. It is easy to verify that the composition * is non-associative and non-commutative but all elements in K^* are idempotent by *, i.e.,

$$a * a = i.$$

PROPOSITION 2. *The set K constitutes a quasi near-ring relative to the addition of the division ring K and the multiplication* *.

Proof. In fact, the right (but not left) distributive law

$$(a + b) * c = a * c + b * c$$

holds and * is non-associative.

Definition 2. Let now K be a division ring and \$ a operation defined by

$$a \,\$\, b = b^{n}\cdot a\cdot b^{1-n}$$

* We acknowledge the referee's comments on a previous draft of this paper.

with $b \neq 0$, and $n > 1$, an integer.

We note that \$ is also non-associative and non-commutative. It is easy to see that the right (but not left) distributive law holds.

PROPOSITION 3. *If K is non-commutative, the set K is a quasi loop relative to the operation* \$.

Proof. This is so, for although i is a bilateral unity and the equation $y \$ a = b$ always has unique solution for every $a \neq 0$, the equation $a \$ x = b$ has no solution in K, because it leads to the equation

$$x^n \cdot a - b \cdot x^{n-1} = 0$$

which admits non trivial solution only for particular elements of a division ring.

Remark. If K is a field, the set K is an abelian group relative to \$ and this group is isomorphic to the multiplicative group of the field K. For, in this case, $a \$ b = a \cdot b$ and thus the equations $a \$ x = b$ and $y \$ a = b$ have unique solutions $x = b * a = y$.

Examples. Models of such structures are, for example, the set M of square non-singular matrices. In particular, the quasi-loop defined in Proposition 3 by the operation \$ may be obtained from the sub-set of M formed by commuting matrices (the kernel of M relative to the ordinary product of matrices).

References

1. Bruck, R. H. Contribution to the theory of loops. Trans. Amer. Math. Soc., **60** (1946), p. 245–354.

2. Blackett, D. W. Simple and semi-simple near-rings. Proc. Amer. Math. Soc., **4** (1953), p. 772–785.

A NOTE ON THE ROTATION OF AXES

PETER HAGIS, JR., Temple University

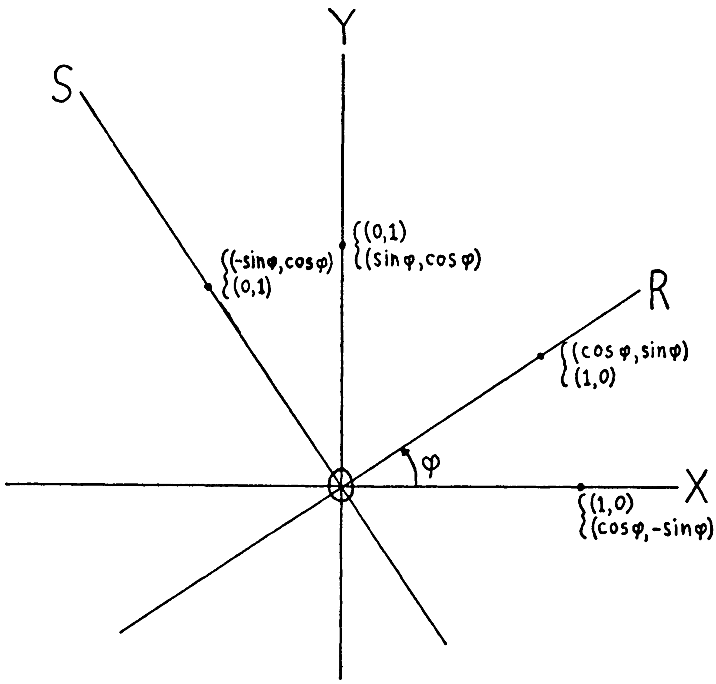
Consider two rectangular coordinate systems, an XY-system and an RS-system, with a common origin O. Let the angle XOR be ϕ. We wish to determine the relations between (x, y) and (r, s), the coordinates of a point with respect to the two systems. The usual procedures for obtaining the transformation equations involve the construction of some right triangles, the use of the addition formulas for the sine and cosine, and some elementary algebra. The following argument involves no constructions, a minimum of trigonometry, and very little algebra.

Let the desired relations be written as $x = f(r, s)$, $y = g(r, s)$. Then the equations $x = h$, $y = k$ become $f(r, s) = h$, $g(t, s) = k$ in the RS-system. Since these equations represent straight lines we see that f and g are both linear functions of r and s. That is, $x = Ar + Bs + C$ and $y = ar + bs + c$, where A, B, C, a, b, c are constants. Since the two systems have a common origin we have immediately $C = c = 0$. From elementary trigonometry the point $(1, 0)$ in the RS-system is the point $(\cos \phi, \sin \phi)$ in the XY-system. Thus, $A = \cos \phi$, $a = \sin \phi$. $(0, 1)$ in the RS-system is $(\cos (90° + \phi), \sin (90° + \phi)) = (-\sin \phi, \cos \phi)$ in the XY-system. Therefore, $B = -\sin \phi$, $b = \cos \phi$. We conclude that

$$\begin{cases} x = r\cos\phi - s\sin\phi, \\ y = r\sin\phi + s\cos\phi. \end{cases}$$

Similarly, $r = Dx + Ey$, $s = dx + ey$. The point (1, 0) in the XY-system has coordinates $(\cos(-\phi), \sin(-\phi)) = (\cos\phi, -\sin\phi)$ in the RS-system, while (0, 1) has coordinates $(\cos(90° - \phi), \sin(90° - \phi)) = (\sin\phi, \cos\phi)$. It follows that

$$\begin{cases} r = x\cos\phi + y\sin\phi, \\ s = -x\sin\phi + y\cos\phi. \end{cases}$$



COMMENTS ON PAPERS AND BOOKS

EDITED BY HOLBROOK M. MACNEILLE, Case Institute of Technology

This department will present comments on papers published in the MATHEMATICS MAGAZINE, lists of new books, and reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent to Holbrook M. MacNeille, Department of Mathematics, Case Institute of Technology, Cleveland 6, Ohio.

ON REPRESENTATION BY A CUBE

J. A. H. HUNTER, Toronto, Ontario

In the Sept.-Oct. 1962 issue of MATHEMATICS MAGAZINE (p. 217) Sam Sesskin conjectured that the sum of the first n natural numbers can never be a cube. I think this can be proved to be true, as follows:

Say we have $S_n = n(n+1)/2 = m^3$.

Setting $2n+1 = Y$, $2m = X$, we have $X^3 = Y^2 - 1$.

All possible integral solutions of this are included in $Y+1 = 4a^2bc^3$, $Y-1 = 2ab^2c^3$, whence $abc^3(2a-b) = 1$. But the greatest common factor of $Y+1$, and $Y-1$, is 2; hence $2abc^3 = 2$, so $a = b = c = 1$.

It follows that $Y+1 = 4$, so $Y = 3$ with $X = 2$, and this is the only possible non-zero positive integral solution.

This seems to establish the truth of the conjecture, since we can ignore the one trivial solution for $n = 1$.

MEDIANS AND INCENTERS

MARLOW SHOLANDER, Western Reserve University

The May issue of this journal gave solutions for Problem 462, generalizing $\max(x, y) = 1/2\{|x-y| + x + y\}$ to cover $\max(x, y, z)$. Symmetric solutions can be obtained by averaging permutations of such solutions. A more difficult requirement is that of avoiding absolute values within absolute values. Since $\max(x, y) + \max(y, z) + \max(z, x) = 2\max(x, y, z) + \text{med}(x, y, z)$, the problem reduces to that of finding a suitable formula for the median. It is easy to verify that $\text{med}(x, y, z) = f(x, y, z)$ where $\{|x-y| + |y-z| + |z-x|\} \cdot f(x, y, z) = z|x-y| + x|y-z| + y|z-x|$ and where x, y, z are not all equal.

The last formula remains valid for complex x, y, z when $\text{med}(x, y, z)$ is defined as the incenter. The proof is simplified by judicious use of linear transformations. We note $\alpha f(x, y, z) + \beta = f(\alpha x + \beta, \alpha y + \beta, \alpha z + \beta)$. A similar result holds for $\text{med}(x, y, z)$ since the incenter is preserved under translations, rotations, and expansions. Hence, no generality is lost by taking $x = 0$, $y = 1$, and $z = b(\cos\theta + i\sin\theta)$. Indeed, the formula is not disturbed by cyclic permutations of x, y, z so in order to complete the proof it is sufficient to show $f(x, y, z)$ lies on the bisector passing through x. Let $|z-1| = a$. Then

$$f(x, y, z) = \frac{b + z}{b + a + 1} = \frac{b(1 + \cos\theta) + bi \sin\theta}{b + a + 1}$$

and

$$\arg f(x, y, z) = \tan^{-1} \frac{\sin\theta}{1 + \cos\theta} = \theta/2.$$

It is to be expected that the orthocenter, orth(x, y, z), and the circumcenter, circ(x, y, z), can also be expressed as weighted means of x, y, and z where the weights are homogeneous functions of sides $a = |y - z|$, $b = |z - x|$, $c = |x - y|$ and where these functions cyclically permute into one another. Reasoning similar to the above establishes that

$$\text{orth}(x, y, z) = \frac{xA + yB + zC}{A + B + C}$$

and

$$\text{circ}(x, y, z) = \frac{x\alpha + y\beta + z\gamma}{\alpha + \beta + \gamma},$$

where $A = a^4 - (b^2 - c^2)^2$ etc., $\alpha = a^2(b^2 + c^2 - a^2)$ etc., and $A + B + C = \alpha + \beta + \gamma = -a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2$. From Heron's Formula, the last expression equals 16 times the square of the area of the triangle. The Euler Line relationship, 2 circ(x, y, z) + orth$(x, y, z) = x + y + z$, is an immediate consequence of the relation $2\alpha = B + C$ and its permutations.

BOOK REVIEWS

Algèbre et Analyse—Exercises By G. Lefort (*Exercises in Algebra and Analysis*). Dunod, Paris, 400 p., 1961, 44 N.F.

This problem book in Algebra and Analysis contains an excellent collection of choice problems (exercises) covering a course in "Mathématiques Générales" taught in French technical colleges and universities.

The book consists of eight parts (chapters) with the following titles:

1. Modern Algebraic concepts and fundamental Algebraic Structures.
2. Complex Numbers. Rational Fractions and Polynomials.
3. Linear and Multilinear Algebra.
4. Functions of a Real Variable. Continuity and Derivative. Sequences of Functions and Numerical Sequences.
5. Integration.
6. Curvilinear Integrals. Double and Triple Integrals.
7. Numerical Series. Series of Functions.
8. Differential Equations. Systems of Differential Equations.

Each chapter consists of three parts. First the general principles and theorems are stated and the definitions are given, then the exercises are enunciated, followed by their complete solutions.

The purpose of these exercises is to test the real understanding by the student of the basic concepts and fundamental theorems of modern Algebra and Analysis. They are intended in particular for students preparing for competitive examinations to enter French Universities and Institutes of Technology.

In American Colleges and Universities, this book will be very useful to those undergraduates (Juniors and Seniors) who are majoring in pure Mathematics and who intend to continue their studies in Graduate Schools.

SOUREN BABIKIAN
Los Angeles City College

A Book of Curves. By E. H. Lockwood. Cambridge University Press, New York, 1961, xi+199 pages, hard cover, $4.95.

The first part of the book deals with fourteen special curves—parabola, ellipse, hyperbola, cardioid, limacon, astroid, nephroid, deltoid, cycloid, right strophoid, equiangular spiral, lemniscate of Bernoulli, tractrix, and catenary. In each case the discussion starts with methods of drawing the curve, leading into a geometric treatment of the properties of the curve, its equations, and a well-organized summary. Excellent drawings make the development easy to follow.

In the second part of the book, some individual curves are briefly discussed, but the main concern is with methods by which new curves can be found. Conchoids, cissoids, strophoids, roulettes, pedal curves, negative pedals, glissettes, evolutes, involutes, spirals and caustic curves are dealt with. Inversion and bipolar coordinates are considered.

A short but well-chosen bibliography and a clear glossary precede two complete indexes, one of names, the other of subjects.

The book is thoughtfully assembled and is skillfully constructed. It should interest readers at all levels of mathematical erudition. Each topic has been gradually developed with this object in mind. It should be a source of stimulation for the better high school students, fine supplementary reading for college mathematics students, and a useful reference book for draftsmen and engineers. It would be a proud possession of any private or public library.

CHARLES W. TRIGG
Los Angeles City College

Fallacies in Mathematics. By E. A. Maxwell. Cambridge University Press, New York, 1959, 95 pp., hard cover, $2.95.

The announced purpose of this little gem is to instruct through entertainment. The author classifies mathematical errors into MISTAKES, HOWLERS which lead innocently to correct results, and FALLACIES which lead by guile to wrong but plausible conclusions. The fallacies of the isosceles triangle, the right angle, the trapezium and the empty circle are analyzed in depth. Then some unusual fallacies in algebra, geometry, differentiation, integration, circular points at infinity, and limits are discussed. The final chapter presents a collection of amazing howlers. Recommended to all who wish to test their logical processes and particularly to teachers who wish additional insight into the sources of student errors.

CHARLES W. TRIGG
Los Angeles City College

Analytical Quadrics. By Barry Spain. Pergamon Press, New York, 1960, ix+135 pp., hard cover, $5.50.

This concise introduction to the analytical geometry of three dimensions is a companion volume of Spain's *Analytical Conics.* The theory of the plane, straight line, sphere, cone, cylinder, central quadrics and paraboloids in standard forms is first developed. Then, homogeneous Cartesian coordinates are used to introduce the plane at infinity. Next, properties of the general quadric, invariants, foci, confocals and linear systems are discussed. Finally, an introductory treatment is presented of plane-coordinates leading to the general treatment of confocal systems. Matrices are touched upon in the first Appendix. Answers to the numerous examples are given at the ends of the respective chapters, and many solutions and hints for solution are given in the second Appendix. A very brief index follows.

CHARLES W. TRIGG
Los Angeles City College

Scarne's Complete Guide to Gambling. By John Scarne. Simon and Schuster, New York, 1961, xx+714 pages, hard cover, $10.00.

The author is a magician, mathematician, gambling expert, and inventor of board games. This volume is the distillation of his lifetime experience in investigating gambling and in exposing crooked practices. A comprehensive treatment is given to various card and dice games, roulette, bingo, the numbers game, lotteries, sweepstakes, horse racing, slot machines, the match game, casino side games, carnival games, punchboards, chain letters, and private betting propositions. In each case, the rules of the game are given, probabilities of winning are mathematically analyzed, and cheating procedures are exposed. The easily read text is enlivened with many anecdotes and numerous pictures. An extensive glossary and excellent index are provided. This book should be an effective antidote for gambling fever. It should convince the naive and the young that the "percentage always favors the house." In the classroom it will add a voice of authority to the mathematics teachers' discussions of probability.

CHARLES W. TRIGG
Los Angeles City College

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

PROPOSALS

509. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$\begin{array}{r} U\ N\ I\ T\ E\ D \\ S\ T\ A\ T\ E\ S \\ \hline A\ M\ E\ R\ I\ C\ A \end{array}$$

in the base 11, introducing the digit α.

510. *Proposed by Miltiades S. Demos, Drexel Institute of Technology.*

Evaluate

$$\prod_{k=1}^{n} \sin\left(\frac{2k-1}{2n}\cdot\frac{\pi}{2}\right).$$

511. *Proposed by Maxey Brooke, Sweeny, Texas.*

With a silver dollar, trace a circle. Choose a point on this circle. Using only the silver dollar and a pencil, construct a circle through the point tangent to the first circle. Assume that you can trace a circle through any two points less than a dollar's diameter apart.

512. *Proposed by R. N. Karnawat and J. M. Gandhi, Government College, Bhilwara, Rajasthan, India.*

Prove the following identity:

$$\binom{n}{2}1^x - \binom{n}{3}2^x + \cdots (-1)^n\binom{n}{n}(n-1)^x = (-1)^{x+1}, \text{ for } n > x,$$

or $$= (-1)^x[n! - 1], \text{ for } n = x.$$

513. *Proposed by Leon Bankoff, Los Angeles, California.*

DE is a chord of a circle O perpendicular to the diameter AB at C. A circle J is inscribed in the space bounded by AC, CE and the arc AE. Show that DJ cuts AC at the point where AC touches the incircle I of triangle ACD.

514. *Proposed by Joseph W. Andrushkiw, Seton Hall University.*

Show that if $f(z)$ is an odd function integrable on $[-1, 1]$, then

$$\int_0^{2k\pi} x^2 f(\sin x)dx = -2k^2\pi^2 \int_0^{\pi} f(\sin x)dx$$

and apply it to evaluate

$$\int_0^{2\pi} \frac{x^2 \sin x}{1 + \cos^2 x}\, dx.$$

515. *Proposed by R. J. Mansfield, Research Council of Alberta, Canada.*

Find an explicit expression for the nth term in the sequence:

$$1,\ 1,\ 1+i,\ 1+2i,\ 3i, \cdots$$

SOLUTIONS

Cubed Primes

488. [September 1962]. *Proposed by Josef Andersson, Vaxholm, Sweden.*

For what values of n is the sum of n integral cubed primes a square (n positive and odd)?

Solution by the proposer.

We know that

$$1^3 + 2^3 + \cdots + (2n)^3 = \left[\frac{2n}{2}(1 + 2n)\right]^2 = n^2(1+2n)^2 \text{ and}$$

$$2^3 + 4^3 + \cdots + (2n)^3 = 8\left[\frac{n}{2}(1 + n)\right]^2 = 2n^2(1+n^2)$$

after subtraction, we have

$$1^3 + 3^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$$

Therefore we have to solve

$$2x^2 - 1 = y^2 \text{ in positive integers.}$$

The fundamental solution being (1, 1), Pell's theory of equations gives

$$n = x_r = [(\sqrt{2}+1)^{2r+1} + (\sqrt{2}-1)^{2r+1}]/2\sqrt{2}$$

$$y_r = [(\sqrt{2}+1)^{2r+1} - (\sqrt{2}-1)^{2r+1}]/2, \qquad r = 0, 1, 2, \cdots$$

In addition we have the formulas

$$x_{r+1} = 3x_r + 2y_r, \qquad y_{r+1} = 4x_r + 3y_r$$

and

$$x_{r+2} = 6x_{r+1} - x_r$$

valid for y_r's also. Thus, the simplest solutions are (1, 1), (5, 7), (29, 41), (161, 239) etc.

One incorrect solution was received.

An Archimedean Theorem

489. [September 1962] *Proposed by Dewey Duncan, East Los Angeles College, California.*

In Archimedes' most prized theorem which was depicted on his tombstone [described by Cicero when serving in Sicily], the common ratio of total areas and volumes of a right circular cylinder to the corresponding entities of the inscribed sphere is 3/2. Replace the cylinder by a right circular cone having its vertex two-thirds of a right angle. Show that the ratio arising here is $(3/2)^2$.

Solution by Joseph B. Bohac, St. Louis, Missouri.

I believe there is an error in the wording of the problem, however if the vertex angle of the cone is changed to 60° the problem is easily solved as follows:

If we denote the slant height of the cone by a, we find by direct computation that the altitude is $a\sqrt{3}/2$ and the radius of the inscribed sphere $r=a\sqrt{3}/6$.

Also the volume of the inscribed sphere $=\pi a^3\sqrt{3}/54$ and volume of cone $=\pi a^3\sqrt{3}/4$.

The surface of the inscribed sphere $=\pi a^2/3$ and surface of cone $=3\pi a^2/4$.

So that,

$$\frac{\text{Volume sphere}}{\text{Volume cone}} = \frac{\text{Surface sphere}}{\text{Surface cone}} = (2/3)^2$$

Those correcting the statement of the problem and submitting solutions are Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; Maxey Brooke, Sweeny, Texas; Brother Christopher Mark, St. Mary's College, California; Michael J. Pascual, Watervliet Arsenal, New York; C. W. Trigg, Los Angeles City College; and the proposer.

From Kent to Ghent to Trent

490. [September 1962]. *Proposed by David L. Silverman, Beverly Hills, California.*

There are twice as many routes from Kent to Ghent, including those through Trent, as there are from Kent to Trent, including those through Ghent. On the other hand, there are twice as many direct routes from Ghent to Trent as there are from Kent to Ghent. How many direct routes connect each pair of towns?

Solution by John W. Moon, University College London.

The problem is equivalent to solving, in positive integers, the simultaneous equations

$$x + zy = 2(z + xy)$$

and

$$y = 2x,$$

where x, y, and z denote the number of direct routes joining Kent and Ghent, Ghent and Trent, and Trent and Kent, respectively. If we substitute the second equation in the first and apply the quadratic formula we find that

$$x = \tfrac{1}{8}[(1 + 2z) \pm \sqrt{4z^2 - 28z + 1}].$$

The quantity under the radical will be the square of a positive odd integer, $2k-1$, if

$$k(k - 1) = z(z - 7).$$

By considering separately the cases $z<k$, $k\leqq z\leqq k+7$, and $k+7<z$, we find that the only solutions to this equation in positive integers are $(k, z)=(1, 7)$ and $(6, 10)$. This implies that the only nontrivial solutions to the original problem are

$$(x, y, z) = (2, 4, 7) \quad \text{and} \quad (4, 8, 10).$$

Also solved by J. L. Brown, Jr., Pennsylvania State University; Daniel I. A. Cohen, Midwood High School, Brooklyn, New York; Monte Dernham, San Francisco, California; Charles E. Franti, Berkeley, California; Jay Gottesfeld, AVCO Rad; J. A. H. Hunter, Toronto, Ontario, Canada; Gilbert Labelle, University of Montreal; Prasert Na Nagara, College of Agriculture, Bangkok, Thailand; P. R. Noland, Department of Education, Dublin, Ireland; Banson Sing, Bethel College, Kansas; C. W. Trigg, Los Angeles City College; and the proposer.

A number of solvers identified only one of the solutions: *Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; Allen W. Brunson, Fenn College, Ohio; W. W. Funkenbusch, Michigan College of Mining and Technology; Brother Christopher Mark, St. Mary's College, California; Jerry L. Pietenpol, Columbia University; Peter Saecker, Science Research Associates, Inc., Chicago; and Ralph N. Vawter, St. Mary's College, California.*

Related In-Circles

491. [September 1962]. *Proposed by C. D. Smith, University of Alabama.*

Given the isosceles triangle with sides a, a, b. Draw altitude h to side b. The in-radius of the given triangle is r, and r_1 is the in-radius of each triangle formed by h. Prove that the in-circle (r) is greater than, equal to, or less than the sum of the two circles (r_1) when $b\lesseqgtr a\sqrt{2}$.

Solution by Leon Bankoff, Los Angeles, California.

When $b\lesseqgtr a\sqrt{2}$, we have $a\gtreqless h\sqrt{2}$ and $b\gtreqless 2h$. Using the relation area = in-radius times semiperimeter, we get $r=hb/(2a+b)$ and $r_1=hb/(2a+2h+b)$. It is now sufficient to show that $(2a+b)\sqrt{2}\gtreqless 2a+2h+b$.

Since $b\gtreqless 2h$, we may write $2h(1-\sqrt{2})\gtreqless b(1-\sqrt{2})$. Dividing throughout by $\sqrt{2}$ and adding $h\sqrt{2}$ to each side, we get $2h\sqrt{2}-2h\gtreqless h\sqrt{2}+b/\sqrt{2}-b$. Then, since $a\sqrt{2} \lesseqgtr 2h$, we have

$$a\sqrt{2}(\sqrt{2}-1) \lesseqgtr 2h(\sqrt{2}-1) \lesseqgtr h\sqrt{2}+b/\sqrt{2}-b, \text{ or}$$

$$(2a+b)\sqrt{2} \lesseqgtr 2a+2h+b, \text{ as required.}$$

Note. The solution for the case $b=a\sqrt{2}$ is almost trivial when we consider that h divides the resulting isosceles right triangle into two equal isosceles right triangles. Since the ratio of in-circle-to-triangle areas is constant in similar triangles, it follows that $\pi r^2=2\pi r_1^2$.

Also solved by Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; Jay Gottesfeld, AVCO Rad; Win Myint, Watervliet Arsenal, New York; H. S. Wilson, Jacksonville University, Florida; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer.

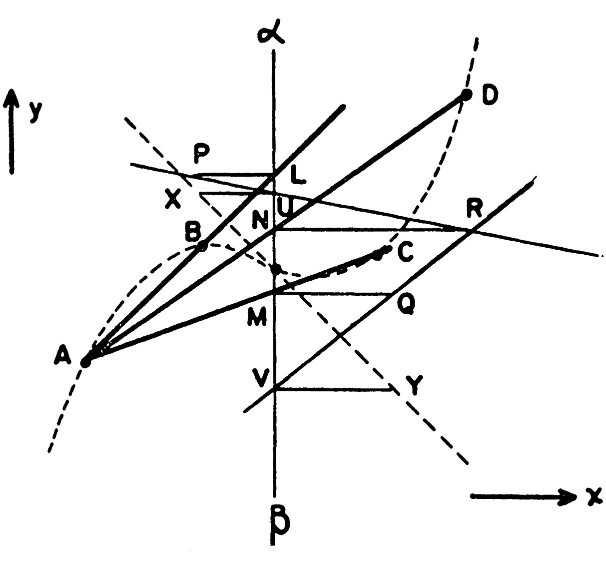
An Interpolation Construction

492. [September 1962]. *Proposed by Ronald Butler, University of Saskatchewan.*

Given n points in a cartesian coordinate plane. Obtain a ruler and compass construction for determining a point, of arbitrary given abscissa, lying on the $(n-1)$st degree curve through the given points.

Solution by the proposer.

The geometrical analogues of either Aitken's or Neville's schemes of iterative linear interpolation provide solutions. Consider Aitken's method and take 4 points A, B, C, D with $\alpha\beta$ any arbitrary ordinate line. Join *any* of the given points to the other three. Thus, using A, AB, AC, and AD are drawn cutting $\alpha\beta$ in L, M, and N respectively. Construct LP, MQ, NR at right angles to $\alpha\beta$, P, Q, and R being on the ordinates through B, C, and D respectively. Join *any*



one of P, Q, and R to the other two. In the figure RP and RQ are drawn and cut $\alpha\beta$ in U and V respectively. Normals to $\alpha\beta$ are constructed at U and V to meet the ordinates PB and QC in X and Y respectively. The join of X and Y cuts $\alpha\beta$ in the required point.

In the Figure the cubic curve through $ABCD$ is drawn for interest. Ref. A. C. Aitken, "On interpolation by proportional parts, without the use of differences." Proc. Edin. Maths. Soc. (2), 3, (1932) 56.

Mostly Composite

493. [September 1962]. *Proposed by Andrezej Makowski, Warsaw, Poland.*

Prove that n^4+4^n $(n=1, 2, 3, \cdots)$ is a prime number only for $n=1$.

I. *Solution by James Shneer, Bellflower, California.*

Let us examine the units digit of all numbers of the form n^4+4^n. It is clear that there are exactly ten distinct cases to consider. These ten are enumerated in the table below.

n, *units digit*	n^4, *units digit*	4^n, *units digit*	n^4+4^n *units digit*
0	0	0	0
1	1	4	5
2	6	6	2
3	1	4	5
4	6	6	2
5	5	4	0
6	6	6	2
7	1	4	5
8	6	6	2
9	1	4	5

From the column at the right we see that every number of the required form "ends" in either 2, 5, or 0 and hence is divisible by either 2 or 5. Therefore only the number 5 itself (corresponding to $n=1$) is both of the form n^4+4^n and prime.

II. *Solution by Wallace Kodis, Cabrillo College, California.*

If n is even, then n^4+4^n has a proper factor of 4 since n will be of the form $2k(k=1, 2, 3, \cdots)$ and

$$(2k)^4 + 4^{2k} = 4(4k^4 + 4^{2k-1}) \quad \text{where} \quad 4k^4 + 4^{2k-1} > 1 \qquad \text{for all } k \geqq 1$$

If n is odd and not 1, then n^4+4^n has a proper factor of $[(2k+1)^2+2^{2k+1} - (2k+1)2^{k+1}]$ for n will be of the form $2k+1(k=1, 2, 3, \cdots)$ and

$$(2k + 1)^4 + 4^{2k+1} = [(2k + 1)^2 + 2^{2k+1} - (2k + 1)2^{k+1}] \cdot [(2k + 1)^2 + 2^{2k+1} + (2k + 1)2^{k+1}]$$

where

$$\left.\begin{aligned}(2k + 1)^2 + 2^{2k+1} + (2k + 1)2^{k+1} > 1\\ (2k + 1)^2 - 2^{2k+1} - (2k + 1)2^{k+1} > 1\end{aligned}\right\} \quad \text{for all } k \geqq 1$$

Also solved by Brother U. Alfred, St. Mary's College, California; Josef Andersson, Vaxholm, Sweden; L. Carlitz, Duke University; Gilbert Labelle, University of Montreal; Prasert Na Nagara, College of Agriculture, Bangkok, Thailand; Robert W. Prielipp, University of Wisconsin; Jerry L. Pietenpol, Columbia University; Brother Raphael, St. Mary's College, California; David L. Silverman, Beverly Hills, California; Ralph N. Vawter, St. Mary's College, California; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer.

Rational Appropriations

494. [September 1962]. *Proposed by C. W. Trigg, Los Angeles City College.*

What two rational fractions with denominators less than 100 most closely approximate $\sqrt[3]{98}$ by excess and by defect?

Solution by Michael J. Pascual and John Zweig (jointly) Watervliet Arsenal, New York.

The simple continued fraction expansion of $\sqrt[3]{98}$ gives us the approximation (using the 6th convergent)

$$\sqrt[3]{98} \approx 4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{4}}}}}} = \frac{355}{77} < \sqrt[3]{98}$$

A non simple continued fraction expansion of $\sqrt[3]{98}$ gives us (using the 7th convergent)

$$\sqrt[3]{98} \approx 5 + \cfrac{1}{-1 + \cfrac{1}{-1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{-2 + \cfrac{1}{-3 + \cfrac{1}{-5}}}}}}} = \frac{438}{95} > \sqrt[3]{98}$$

And if

$$\frac{355}{77} < \frac{p}{q} < \frac{438}{95}$$

then

$$0 < \frac{p}{q} - \frac{355}{77} < \frac{438}{95} - \frac{355}{77}$$

$$0 < \frac{77p - 355q}{77q} < \frac{1}{(77)(95)}$$

or $0<95(77p-355q)<q$, and since $77p-355q>0$, then $95<q$.

Evaluating the approximations with denominators of 96, 97, 98 and 99, we find none better than the two above.

Also solved by Brother U. Alfred, St. Mary's College, California; Merrill Barneby, University of North Dakota; Dermot A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts; Maxey Brooke, Sweeny, Texas; Sam Kravitz, East Cleveland, Ohio; Sidney Kravitz, Picatinny Arsenal, New Jersey; Prasert Na Nagara, College of Agriculture, Bangkok, Thailand; Jerry L. Pietenpol, Columbia University; Ralph N. Vawter, St. Mary's College, California; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer.

Comment on Problem 455

455. [September 1961 and March 1962]. *Proposed by Leonard Carlitz, Duke University.*

Let $n>1$. Show that:

a) $$x(x+1)\cdots(x+n-1) \equiv x^n - x \pmod{n}$$

if and only if n is prime;

b) $$\prod_{\substack{a=1 \\ (a,n)=1}}^{n} (x+a) \equiv x^{\phi(n)} - 1 \pmod{n}$$

if and only if $n=p$ or $2p$, where p is a prime.

Comment by Josef Andersson, Vaxholm, Sweden.

1. $n=2\cdot 3=6$; $\phi(6)=2$.
   The Product $P\equiv(x+1)(x+5)\equiv x^2-1=x-1 \pmod{\phi(6)}$.
   Correct.
2. $n=2\cdot 5=10$; $\phi(10)=4$.
   $P\equiv(x^2-1)(x^2-9)\equiv x^4-1 \pmod{10}$.
   Correct.
3. $n=2\cdot 7=14$; $\phi(14)=6$.
   $P\equiv(x^2-1)(x^2-9)(x^2-25)\equiv(x^4-10x^2+9)(x^2+3)$
   $\equiv x^6-7x^4+7x^2-1 \pmod{14}$.
   Incorrect.
4. $n=2\cdot 11=22$; $\phi(22)=10$
   We find $\equiv x^{10}+11x^8-11x^2-1 \pmod{22}$.
   Incorrect.
5. $n=2\cdot 17=34$; $\phi(34)=16$.
   $P\equiv x^{16}-1 \pmod{34}$.
   Correct.

But $3=2^{2^0}+1$, $5=2^{2^1}+1$, $17=2^{2^2}+1$, whereas 7 and 11 are not the primes of Fermat-Gauss.

Comment by the proposer.

As pointed out by Andersson the original statement of part b) is incorrect. It should read as follows:

$$\prod_{\substack{a=1\\(a,n)=1}}^{n} (x+a) \equiv x^{\phi(n)} - 1 \pmod{n}$$

if and only if $n=4$ or p where p is an arbitrary prime or $n=2p$, where p is a prime of the form 2^k.

Proof. We have

$$\prod_{\substack{a=1\\(a,2p)=1}}^{2p} (x+a) \equiv (x^{p-1} - 1) \pmod{p}.$$

But $(x+1)^{p-1} \equiv x^{p-1} - 1 \pmod{2}$ if and only if $p-1=2^r$ by a well known property of binomial coefficients.

The remainder of the proof (of the necessity) is carried over without change.

Comment on Problem 468

468. [January and September 1962]. *Proposed by Paul D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

The three perimeter bisecting lines, each of which passes through a corresponding midpoint of a side of a given triangle, meet at a point S. The three perimeter bisecting lines, each of which passes through a corresponding vertex of the given triangle meet in a point N. Show that S is the midpoint of the line segment joining the incenter of the given triangle to the point N.

Comment by Nathan Altshiller Court, University of Oklahoma.

Thomas pointed out an original approach to a figure which abounds in interesting properties, to some of which it may be worth while to call to the attention to the readers of this Magazine.

In the figure accompanying the published solution the point M is a vertex of the medial triangle (T') of the given triangle $(T)=ABC$; the line MM' is parallel to the bisector AT of the angle A of (T), it corresponds therefore to AT in the homothecy which transforms (T) into (T') [Altshiller Court, College Geometry, sec. ed., p. 68, arts. 96–98, 1952]. Hence MM' is the bisector of the angle M of (T') and the point S is the incenter of the triangle (T'), thus

(1) $$IG : GS = 2 : 1,$$

where G denotes the common centroid of the triangles (T), (T').

The point N is known as the Nagel point of (T). In the homothecy (T), (T'), the point which corresponds, in (T'), to the point N of (T) coincides with the incenter S of (T') (ibid., p. 161, art. 337), hence

(2) $$NG : GS = 4 : 1.$$

From (1) and (2) we have

$$NS = IS = 2GS$$

Thus S is the midpoint of IG.

It follows also from (1) and (2) that the segment IS is divided by the points G, N internally and externally in the same ratio 2:1, that is, the two pairs of points I, S and G, N are harmonic. The points G, N are thus the centers of similitude of the inscribed circles (I), (S) of the triangles (T), (T').

Thus the circle (GN) having GN for diameter is the circle of similitude of (I), (S), while the circles (G), (N) centered at G, N and coaxal with (I), (S) are the circles of antisimilitude of the latter two circles.

Let O, H, O_n denote the circumcenter, the orthocenter, and the ninepoint center of (T). The Euler line OGO_nH of (T) meets the line $IGSN$ in G and those two lines are divided into proportional pairs of segments by the point G and the lines OI, O_nS, HN. Hence the latter three lines are parallel to each other. From the three similar triangles GOI, GO_nS, GHN a number of propositions may be derived, as for instance: In a triangle (T) the line drawn through the orthocenter H parallel to the line joining the incenter I to the circumcenter O, passes through the Nagel point N, and $HN = 2 \cdot OI$.

The triangle (T') has (O_n) for its circumcircle. If (T') is taken for the basic triangle, then (T) is its anticomplementary triangle. Since O_nS, HN are parallel lines, we have: The line joining the orthocenter and the Nagel point of the anticomplementary triangle of a given triangle is parallel to the line joining the circumcenter and the incenter of the latter triangle. The ratio of the two segments is equal to 4.

These properties and others may also be obtained by considering the two harmonic sets of points $(IGSN)$ and (OGO_nH) (ibid., p. 104, art. 210) which have the point G in common (cf. ibid., p. 170, art. 357).

Other properties of the Nagel point may be found in William Gallatly, The Modern Geometry of the Triangle, pp. 20–22, arts. 30, 31. Francis Hodgson, London. Sec. ed. R. A. Johnson, Modern Geometry, p. 228 ff. New York, 1929. Reprint, New York, 1960.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 310. Show that $\sin\theta > \tan^2 \theta/2$ for $0 < \theta < \pi/2$. [*Submitted by M. S. Klamkin.*]

Q 311. Erect a perpendicular at a point P on a line. [*Submitted by C. W. Trigg.*]

Q 312. Show that no equilateral triangle which is either inscribed in or circumscribed about an ellipse (excluding the circular case) can have its centroid coincide with the center of the ellipse. [*Submitted by M. S. Klamkin.*]

(Answers on page 108)

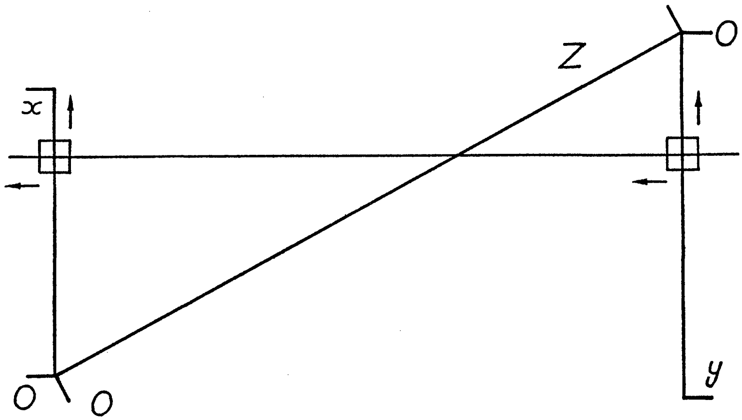


FIG. 3

free to slide across and along the x and y scales. (Imagine two slide rules which have been fixed at right angles by gluing together the upper faces of their cursors.) There would be free joints between the ends of the z scale and the other two scales. So that the transversal could be set for any x and y, it would be necessary for the z scale to be longer than the total length of the other two scales. Beyond this, increase in its length would not affect the accuracy. With such a set of rods the scales would automatically be in the best position for accurate reading in every case.

ANSWERS

A 310. Let $\cos\theta = x$. Then

$$\sqrt{1-x^2} > \frac{1-x}{1+x} \quad \text{or} \quad (1+x)^{3/2} > (1-x)^{1/2}$$

which is obviously true.

A 311. With a convenient radius r and center O describe a circle cutting the line at P and again at A. Draw AO extended to meet the circle again at B. BP is perpendicular to the line at P since angle BPA is a right angle, being inscribed in a semi-circle. This method requires one compass opening and the drawing of one circle and two lines, as opposed to two compass openings, three circles and one line by the more conventional method. If P is at the end of a line segment an additional line would have to be drawn to use the conventional method.

A 312. Orthogonally project the ellipse into a circle. The equilateral inscribed or circumscribed triangles will become inscribed or circumscribed non-equilateral triangles whose centroids cannot coincide with the center of the circle. Since centroids transform into centroids, the proof is completed.

(Quickies on page 142)
